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# Economic Capital Allocation



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## Preface

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Min Wang



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## List of papers

I) Min Wang and Lasse Koskinen, Various Faces of Risk Measures: Internal Model's Perspective, page 1-24, Conference ASTIN (Actuarial Studies In Non-life insurance), 1-4 June 2009, Helsinki,

[www.actuaries.org/ASTIN/Colloquia/Helsinki/Papers/S1\\_29\\_Wang\\_Koskinen.pdf](http://www.actuaries.org/ASTIN/Colloquia/Helsinki/Papers/S1_29_Wang_Koskinen.pdf)

Lasse Koskinen wrote the final version.

II) Min Wang, Capital allocation based on the Tail Covariance Premium Adjusted, Insurance: Mathematics and Economics, Volume 57, July 2014, Pages 125-131, ISSN 0167-6687, (<http://www.sciencedirect.com/science/article/pii/S0167668714000614>)

III) Min Wang, On economic capital allocation based on tails, 10 pages, manuscript, 2013

IV) Min Wang, Upper Tail Covariance of Log-elliptical Distributions, 14 pages, manuscript, 2013

Explanation for the article by co-author:

In I), I worked on optimization under variance premium principle, see section 4.2, which is my main initial contribution. I also wrote the section on risk measure. Lasse Koskinen wrote the final version.



## Summary of included articles

I) Min Wang and Lasse Koskinen, Various Faces of Risk Measures: Internal Model's Perspective, page 1-24, Conference ASTIN (Actuarial Studies In Non-life insurance), 1-4 June 2009, Helsinki

In this article, we consider several aspects of risk measures from the internal models perspective. We critically review the most widely used classes of risk measures. Especially, we attempt to clear up some of the most commonly misconstrued aspects: the choice between risk measures, and practical data and forecasting issues, like the importance of robustness. As a new result, solvency capital requirement is optimized under variance premium principle. The use of tail conditional trimmed mean is proposed as a robust risk estimator. One objective of this article is to emphasize that one single risk measure or a specific axiomatic system is not appropriate for all purposes.

II) Min Wang, Capital allocation based on the Tail Covariance Premium Adjusted, Insurance: Mathematics and Economics, Volume 57, July 2014, Pages 125-131, ISSN 0167-6687  
The current Solvency II process makes risk capital allocation to different business lines more and more important. This paper considers two business lines with the exponential loss distributions linked by a Farlie-Gumbel-Morgenstern (FGM) copula, modelling the dependence between them. As allocation principle we use the Tail Covariance Premium Adjusted (TCPA) and obtain expressions for the allocation to the two business lines.

III) Min Wang, On economic capital allocation based on tails, 10 pages, manuscript, 2013  
In this paper, we introduce a new risk measure, Tail Covariance Premium Adjusted (TCPA). When choosing this risk measure in the proportional capital allocation case, the correlation comes to play a role for capital allocation, but not too large a role as in the case of the covariance capital allocation principle. Tail Covariance Premium Adjusted (TCPA) includes TVaR and the covariance at the tail of the sum of risks.

IV) Min Wang, Upper Tail Covariance of Log-elliptical Distributions, 14 pages, manuscript, 2013

During the financial crisis, significant changes in the insurance and financial markets are

giving increasing attention to the dependence of major losses. Recently, there has been growing interest among insurance and investment experts to focus on the relationship of major claims in insurance or major losses in investment worlds. In this paper we throw light upon the relationship of two claims when both exceed certain thresholds. We introduce the concept of Upper Tail Covariance, a covariance of two claims conditional on both exceeding some thresholds. We investigate the Upper Tail Covariance in log-elliptical cases. The log-elliptical distributions are a family of distributions that include the more familiar log-normal distribution. The class of log-elliptical distributions has been introduced in the applications in insurance and actuarial science. There is a fair amount of discussion of this important class as a tool for modelling risk dependencies. We derive expressions of Upper Tail Covariance for log-elliptical distributions. The theoretical results are illustrated by considering log-normal distributions. Numerical examples illustrate the concept in log-normal settings. The new concept Upper Tail Covariance can be a risk measure or part of capital allocation principles.

## Abstract

For my Licentiate thesis, I conducted research on risk measures. Continuing with this research, I now focus on capital allocation. In the proportional capital allocation principle, the choice of risk measure plays a very important part.

In the chapters Introduction and Basic concepts, we introduce three definitions of economic capital, discuss the purpose of capital allocation, give different viewpoints of capital allocation and present an overview of relevant literature. Risk measures are defined and the concept of coherent risk measure is introduced. Examples of important risk measures are given, e. g., Value at Risk (VaR), Tail Value at Risk (TVaR). We also discuss the implications of dependence and review some important distributions.

In the following chapter on Capital allocation we introduce different principles for allocating capital. We prefer to work with the proportional allocation method.

In the following chapter, Capital allocation based on tails, we focus on insurance business lines with heavy-tailed loss distribution. To emphasize capital allocation based on tails, we define the following risk measures: Conditional Expectation, Upper Tail Covariance and Tail Covariance Premium Adjusted (TCPA).

In the final chapter, called Illustrative case study, we simulate two sets of data with five insurance business lines using Normal copulas and Cauchy copulas. The proportional capital allocation is calculated using TCPA as risk measure. It is compared with the result when VaR is used as risk measure and with covariance capital allocation.

In this thesis, it is emphasized that no single allocation principle is perfect for all purposes. When focusing on the tail of losses, the allocation based on TCPA is a good one, since TCPA in a sense includes features of TVaR and Tail covariance.

## Sammanfattning

Föreliggande avhandling behandlar riskhantering, riskmått och allokering av riskkapital på olika affärsområden inom bolag som är speciellt riskbenägna, t.ex. försäkringsbolag.

I min licentiatavhandling studerade jag begreppet riskmått. I min fortsatta forskning har huvudintresset legat på allokering av kapital. Härvid kan konstateras att valet av specifikt riskmått har en mycket stor betydelse, i synnerhet om man väljer den s.k. proportionella allokeringemetoden.

I kapitlen 1 och 2 introduceras definitionen av ekonomiskt kapital, diskuteras för- och nackdelarna med ett dylikt begrepp ur olika synvinklar samt ges en översikt av relevant litteratur. Riskmått definieras och det viktiga begreppet koherent riskmått diskuteras och exemplifieras. Viktiga riskmått som används i praktisk bank- och försäkringsverksamhet är bl.a. *Value at Risk* (VaR) och *Tail Value at Risk* (TVaR). Problem som sammanhänger med beroende mellan skadetyper diskuteras och ett antal ofta använda sannolikhetsfördelningar presenteras.

I kapitel 3 presenteras olika typer av principer som används för kapitalallokering. I denna avhandling arbetas företrädesvis med den s.k. proportionella allokeringemetoden.

Kapitel 4 behandlar allokeringemetoder baserade på fördelningens ”svans”, dvs. på stora värden på utfallet/skadan. Fokus ligger på försäkringsslag med s.k. tjocksvansade fördelningar. Karakteristiska för dessa är att mycket stora utfall kan förekomma med rätt stor sannolikhet. Riskmått som bygger på fördelningarnas svans är t.ex. *Conditional Expectation*, *Upper Tail Covariance* och *Tail Covariance Premium Adjusted* (TCPA).

I kapitel 5 illustreras teorin med några simulerade fallstudier där beroendeförhållanden mellan olika försäkringsslag uttrycks genom s.k. *copulas*. Proportionella allokeringprincipen med TCPA som riskmått tillämpas. Det fallet jämförs med VaR som riskmått och kovariansallokering. Då tyngdpunkten ligger på extrema utfall är TCPA ett bra verktyg, eftersom den kombinerar drag av Tail Covariance och VaR. Något riskmått som är bäst i alla situationer finns dock inte.



# Chapter 1

## Introduction

### 1.1 Main problem of the research

The main problem of the research is how to allocate economic capital to different business lines. There are many different methods of allocation proposed in research papers. This thesis will put emphasis on capital allocation based on tails, since the major losses by definition are tail events. The thesis will start with the basic concepts of economic capital and allocation.

### 1.2 Economic capital

A bank or an insurance company always faces some risks that could cause a financial loss. Economic capital is the realistic amount of capital that is needed to cover losses at a certain risk tolerance level (Shaw et al. (2010) [51]).

### 1.2.1 What is the economic capital?

In the *Specialty Guide on Economic Capital* [52] (2004), the authors found that there is no one consistent definition of economic capital in use in the marketplace. Definitions in use are numerous, but the following three main definitions, based on 77 responses to EC survey (2002), demonstrate the main themes of the various practical alternatives currently in use.

- Definition 1

Economic capital is defined as sufficient surplus to meet potential negative cash flows and reductions in value of assets or increases in value of liabilities at a given level of risk tolerance, over a specified time horizon.

- Definition 2

Economic capital is defined as the excess of the market value of the assets over the fair value of liabilities required to ensure that obligations can be satisfied at a given level of risk tolerance, over a specified time horizon.

- Definition 3

Economic capital is defined as sufficient surplus to maintain solvency at a given level of risk tolerance, over a specified time horizon.

Valdez (2012) [65] thinks that it is the amount set aside, usually in excess of assets backing all liabilities, so that the company:

- could withstand and absorb "unexpected losses" from all risks it is facing;
- would remain solvent with high probability; and
- is able to cover obligations to its customers as promised.

It captures a wide spectrum of risks such as insurance risk, market risk, credit risk and operational risk, as well as dependencies between them and various other complexities such as transferability of capital, and expresses all of this as a single number.

## 1.2.2 The purpose of economic capital

Calculating the economic capital for a firm has its many purposes, see the list from the *Specialty Guide on Economic Capital* [52] (2004). The list is not intended to be exhaustive, but it does capture the major uses of economic capital in today's insurance industry environment:

1. Determination of the company or product risk profile
2. Capital budgeting
3. Evaluation of required capital in merger and acquisition situations
4. Insurance product pricing
5. Risk tolerances and constants
6. Asset/liability management
7. Calculating Risk-Adjusted Return on Capital
8. Performance Measurement
9. Incentive Compensation
10. Rating agency and regulatory discussions

A company may do an economic capital calculation according to external criteria laid down by the regulators for regulatory capital purposes or other criteria, e.g., to satisfy specific standards prescribed by a rating agency.

Currently the most popular risk measure used in banking and insurance is the one-year 99.5% Value at Risk (VaR). For example, under the UK's Individual Capital Assessment (ICA) regime and Solvency II, an insurance company needs to hold enough capital such that there is a probability of 99.5% of survival over a one-year time horizon, or in other words, the probability of insolvency over 12 months is no more than 0.5%.

Economic capital plays a central role in supervision, product pricing, risk assessment,



risk management and hedging, capital allocation, performance management and financial reporting, see Corrigan et al. (2009) [10].

Economic capital can be calculated by risk measures. This is very important part in risk management.

## 1.3 Capital allocation

Dhaene et al. (2011) [19] and Valdez (2012) [65] give a definition of capital allocation, the term typically referring to the subdivision of the aggregate capital held by the firm across its various constituents, e.g.

- lines of business
- its subsidiaries
- product types within line of business
- types of risks: e.g. market, credit, pricing/underwriting, operational
- territories, e.g. distribution channels

In the slides from Valdez (2012) [65], capital allocation is a very important component of enterprise risk management, such as, identifying, measuring, pricing and controlling risks.

### 1.3.1 Why allocate?

There are many opinions about why to allocate capital. Most agree that allocation to different business lines is a risk management for pricing or performance measurement, see Corrigan et al. (2009) [10], DiCaro (2010) [21] and Dhaene et al. (2011) [19].

Holding economic capital is a cost. The cost needs to be allocated across business lines. At the same time, the allocated cost also makes the return of business lines more clear. That can be used to judge the performance of the different lines.

In Venter (2004) [66], capital allocation is generally not an end in itself, but rather an intermediate step in a decision-making process. In Bodoff (2009) [7], how a firm allocates capital, similar to other cost allocation decisions, can significantly affect the measured profitability of a particular line of business. Moreover, allocating capital can affect target pricing margins and the volume of business the company writes in each line of business and product type. As a result, the topic is critically important and often the subject of contentious debate among the heads of the firm's various business units.

DiCaro (2010) [21] answered why we are allocating capital as follows:

- Determine which business units are most profitable relative to the risk they bring to the enterprise
- Include a risk charge in pricing
- Compensation/performance management
- Regulatory/Rating compliance?
- Enterprise Risk Management (ERM) processes allocate capital to risk categories: catastrophe risk, market risk, counterparty risk ...

### 1.3.2 Different viewpoints

In Dhaene et al. (2011) [19], there are some different viewpoints about capital allocation.

- Owners' i.e. shareholders'  
Allocating capital may help to identify areas of risk consumption within a given organization and support the decision making concerning business expansions, reductions or even eliminations.
- Business line managers'  
A good allocation helps evaluate performance of his own business line and compare with other business lines. It allows one to distinguish the most profitable business lines and hence may assist in remunerating the business line managers.

- Regulators'

The regulator, primarily sharing the interests of depositors and policyholders, establishes rules to determine the required capital to be held by the company. From a capital allocation perspective, the regulators will be concerned to see that capital is optimally used within the business to ensure security for policyholders.

In this thesis, the regulator's viewpoints are used.

### 1.3.3 Overview of literature

Dhaene et al. (2011) [19] give a very good overview of the literature. There are many different approaches to allocate the aggregate capital of a company to its different business units. Mutual dependencies that may exist between the performances of the various business units make capital allocation a non-trivial exercise. Accordingly, there is an extensive amount of literature on this subject with a wide number of proposed capital allocation algorithms. Cummins (2000) [11] provides an overview of several methods suggested for capital allocation in the insurance industry and relates capital allocation to management decision making tools such as RAROC (risk-adjusted return on capital) and EVA (economic value added). Myers and Read Jr. (2001) [44] consider capital allocation principles based on the marginal contribution of each business unit to the company's default option. The default value is the present value of the insurance company's option to default. LeMaire (1984) [42] and Denault (2001) [13] consider that in the language of game theory, the risk capital allocation problem is modelled as a game between the constituents of the firm. The allocation of the overall capital costs to the policies has to be fair, which means that no subportfolio of policies, would be better off on their own. Tasche (2000) [54] calculates the risk contribution by the marginal ('Euler') principle and argues by means of portfolio steering/performance measurement. The risk contribution should be calculated in such a way that it rewards policies with a positive contribution to the overall result, and punishes policies with a negative contribution. Further approaches to capital allocation include Kalkbrenner (2005) [35], where an axiomatic allocation framework is used, formulating desirable properties. Further, there is an extension of this approach to spectral

measures of risk, see Overbeck (2004) [46]. A commentary on the various approaches to allocating capital has appeared in Venter (2004) [66]. Another very general approach to capital allocation using different methods is found in Denneberg and Maass (2006) [14]. A recent work by Kim and Hardy (2008) [38] proposed a method based on an insolvency exchange option (default option) and which explicitly accommodates the notion of limited liability of the shareholders and can further decompose the allocated capital.

Panjer (2001) [48] considers the particular case of multivariate normally distributed risks and provides an explicit expression of marginal cost based allocations, when the risk measure used is Tail Value at Risk (TVaR). Landsman and Valdez (2003) [41] extends these explicit capital allocation formulas to the case where risks belong to the class of multivariate elliptical distributions, for which the class of multivariate normal is a special case. Dhaene et al. (2008) [16] derive the results of Landsman and Valdez (2003) [41] in a rather more straightforward manner and apply these to sums that involve normal as well as lognormal risks. In Valdez and Chernih (2003) [60], expressions for covariance-based allocations are derived for multivariate elliptical risks. Tsanakas (2004) [55] studies allocations where the relevant risk measure belongs to the class of distortion risk measures, while Tsanakas (2008) [57] extends these allocation principles to the more general class of convex risk measures including the exponential risk measures. Furman and Zitikis (2008) [26] introduce the class of weighted risk capital allocations "which stems from the weighted premium calculation principle".

The multitude of allocation methods proposed in the literature is complicated. Allocation methods are sometimes proposed in an *ad hoc* fashion usually lacking much economic justification and are thereby viewed as arbitrary. This motivated some authors to doubt the legitimate purpose of the exercise itself of allocating capital e.g. Gründl and Schmeiser (2007) [29]. Gründl and Schmeiser (2007) [29] point out this importance because accordingly, certain allocation techniques can dangerously lead to wrong financial decisions. For example, they think that that capital allocation to lines of business based on the Myers and Read approach is either not necessary for insurance rate making (in the case of no frictional costs) or even leads to incorrect loadings (when frictional costs are considered). The following references are taken from the literature list in Valdez (2012) [65].

Good overview of methods:

- Cummins (2000) [11]; Venter (2004) [66]

Some methods based on decision making tools:

- Cummins (2000) [11] - Risk-adjusted return on capital (RAROC), economic value added (EVA)
- LeMaire (1984) [42] and Denault (2001) [13] - game theory
- Tasche (2000) [54] - marginal costs
- Kim and Hardy (2008) [38] - solvency exchange option with limited liability

Some methods based on risk measures/distributions:

- Panjer (2001) [48] - TVaR, multivariate normal
- Landsman and Valdez (2003) [41] - TVaR, multivariate elliptical
- Valdez and Chernih (2003) [60] - covariance-based allocation, multivariate elliptical
- Tsanakas (2004) [55] and (2008) [57] - distortion risk measures, convex risk measures
- Furman and Zitikis (2008) [26] - weighted risk capital allocation

Methods also based on an optimality principle:

- Dhaene et al. (2003) [20]; Laeven and Goovaerts (2004) [39]; Zaks, Forstig and Levikson (2006) [24]
- Biard (2010) [6] study the optimal allocation to minimize the asymptotic ruin probability.

Methods special emphasis on heavy-tailed distribution

- Hult and Lindskog (2006) [32] analyze the impact of rules for transfer of capital on the ruin probability and they draw conclusions about possible benefits from diversification;
- Asimit et al. (2011) [3] - TVaR, distributions from Maximum Domain of Attraction (MDA)
- Asimit et ai. (2013) [4] - the distortion and weighted risk measures and allocations, as well as their special cases such as the conditional layer expectation, tail value at risk, and the truncated tail value at risk, multivariate Pareto distribution of the second kind



# Chapter 2

## Basic concepts

### 2.1 Risk measures

Economic capital and capital allocation are calculated based on risk measures. An introduction to the theory of risk measures may be found in Wang (2009) [67].

According to Hardy (2006) [30], in actuarial applications we often work with loss distributions for insurance products. For example, in Property & Casualty insurance, we can develop a compound Poisson model for the losses under a single policy or a whole portfolio of policies. In life insurance, we can develop a loss distribution for a portfolio of policies, often by stochastic simulation.

In addition, it is usually appropriate to assume, in insurance contexts, that the loss  $X$  is non-negative. It is not essential however, and the risk measures that we describe can be applied (perhaps after some adaptation) to random variables with possible values in any part of the real line.

Following Dhaene et al. (2008) [17], we consider a set  $\Gamma$  of real-valued random variables defined on a given measurable space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We will assume that  $X, Y \in \Gamma$  implies that  $X + Y \in \Gamma$ , and also  $aX \in \Gamma$  for any  $a > 0$  and  $X + b \in \Gamma$  for any real  $b$ . A functional

$$\rho : \Gamma \rightarrow \mathbb{R},$$



mapping every element of a loss (or profit) distribution in  $\Gamma$  to the real numbers, is called a *risk measure* (with domain  $\Gamma$ ). The risk measure is assumed in some way to encapsulate the risk associated with a loss distribution.

In this thesis, we will interpret  $\Omega$  as the set of all possible states of nature at the end of some fixed reference period, for instance one year. The set  $\Gamma$  will be interpreted as the extended set of financial losses under consideration at the end of the reference period, related to insurance and investment portfolios that a particular regulatory authority controls.

Let  $X$  be an element of  $\Gamma$ . In case all claims of the corresponding insurance and investment portfolio are settled at the end of the reference period and all premiums are paid at the beginning of the reference period, the (aggregate) loss  $X$  can be defined as claims minus the sum of premiums and investment income. In a general setting, we can define  $X$  as the sum of the claims to be paid out over the reference period and the provisions to be set up at the end of the reference period, minus the sum of the provisions available at the beginning of the reference period, the investment income, and the premiums received over the reference period. Here, claims, premiums and provisions are understood as gross amounts, i.e., including expenses. The valuation principles, on the basis of which the value of the assets (represented by the provisions available, the premiums received and the investment income generated) and in particular the liabilities (represented by the provisions to be set up and the claims to be paid out) are determined, are left unspecified in this thesis; our set-up is compatible with any particular valuation basis.

### 2.1.1 Coherent risk measure

Corrigan et al. (2009) [10] agreed that a good allocation method should be coherent.

In Artzner et al. (1999) [2], a risk measure  $\rho$  is called a coherent risk measure, if it satisfies the following axioms: *monotonicity, positive homogeneity, translation invariance and subadditivity*.

**Axiom 2.1.** *Monotonicity: for any  $X$  and  $Y \in \Gamma$  with  $X \leq Y$ , we have  $\rho[X] \leq \rho[Y]$ .*

This rules out the risk measure,  $\rho[X] = E[X] + \alpha\sigma[X]$ , where  $\alpha > 0$  and  $\sigma$  denotes the standard deviation operator.

**Axiom 2.2.** *Positive homogeneity: for any  $\lambda > 0$  and  $X \in \Gamma$ ,  $\rho[\lambda X] = \lambda\rho[X]$ .*

If position size directly influences risk (for example, if positions are large enough that the time required to liquidate them depends on their sizes) then we should consider the consequences of a lack of liquidity when computing the future net worth of a position.

**Axiom 2.3.** *Translation invariance: for any  $X \in \Gamma$  and all real numbers  $b$ , we have  $\rho[X + b] = \rho[X] + b$ .*

This says that a sure loss of amount  $b$  simply increases the risk by  $b$  and it is an axiom for accounting-based risk measures. For many external risk measures, such as a margin deposit, the accounting-based risk measures seem to be reasonable. For internal risk measures, attitude-based measures may be preferred.

**Axiom 2.4.** *Subadditivity: for all  $X$  and  $Y \in \Gamma$ ,  $\rho[X + Y] \leq \rho[X] + \rho[Y]$ .*

We argue that a subadditivity property, which reflects the diversification of portfolios (see Meyers (2000) [43]), or that ‘a merger does not create extra risk,’ is a natural requirement. In the following, risk measure can also be a function of a vector. For example, if  $\mathbf{X} = (X_1, \dots, X_n)^T$ ,  $\rho[\mathbf{X}] = (\rho[X_1], \dots, \rho[X_n])^T$ .

**Note 1.** *Example of non-coherent risk measure, Value at Risk*

*It is well known that value at risk is not, in general, a coherent risk measure as it does not respect the sub-additivity property, see the example in the next section. An immediate consequence is that value at risk might discourage diversification.*

*Value at risk is, however, coherent, under the assumption of elliptically distributed losses (e.g. normally distributed).*

*Comparing different risk measures, Value at Risk is very much used and robust.*

## 2.1.2 Important risk measures

### 2.1.2.1 Value at Risk - the quantile risk measure

According to Hardy (2006) [30], the Value at Risk, or VaR risk measure was actually in use by actuaries long before it was reinvented for investment banking. In actuarial contexts it is known as the quantile risk measure or quantile premium principle. VaR is always specified with a given confidence level  $\alpha$  - typically  $\alpha = 95\%$  or  $99\%$ .

In broad terms, the  $\alpha$ -VaR represents the loss that, with probability  $\alpha$ , will not be exceeded. Since that may not define a unique value, for example, if there is a probability mass around the value, we define the  $\alpha$ -VaR more specifically.

**Definition 2.1.** *For a given probability level  $\alpha$ , Value at Risk (VaR) is denoted by  $Q_\alpha[X]$ , or alternatively  $VaR_\alpha[X]$ , and defined as the  $\alpha$ -quantile of  $X$ , i.e.,*

$$Q_\alpha[X] = \inf\{x \in \mathbb{R} \mid \Pr[X \leq x] \geq \alpha\} \quad \text{for } \alpha \in (0, 1). \quad (2.1)$$

For continuous distributions this simplifies to  $Q_\alpha[X]$  such that

$$\Pr[X \leq Q_\alpha] = \alpha. \quad (2.2)$$

That is,  $Q_\alpha[X] = F_X^{-1}(\alpha)$  if  $F_X(\alpha)$  is continuous and strictly monotone, where  $F_X(\alpha)$  is the cumulative distribution function of the loss random variable  $X$ . The reason for the 'inf' term in the Definition 2.1 is that for loss random variables that are discrete or mixed continuous and discrete, we may not have a value that exactly matches equation (2.2).

The following lemma expresses the quantiles of a function of a random variable in terms of the quantiles of the random variable.

**Lemma 2.5. Quantiles of transformed random variables** *Let  $X$  be a real-valued random variable, and  $\alpha \in (0, 1)$ . For any non-decreasing and left continuous function  $g$ , it holds that*

$$Q_\alpha[g(X)] = g(Q_\alpha[X]).$$

A proof of this result can be found in Dhaene et al. (2002) [15].

**Example 2.6.** *It is very easy to create an example of a violation of VaR subadditivity: for example, consider two different bonds A and B with nonoverlapping default probabilities (if one defaults the other will not and vice versa). A portfolio that contains both bonds may have a global VaR which is greater than the sum of the two VaRs. For instance, consider the following numerical example. Each bond has a default with a probability of 4%, and we receive recovery value at 70 if a default occurs. Otherwise they will redeem at 100. The 95% VaR of each bond is therefore 70, so  $VaR_{95\%}[A] = VaR_{95\%}[B] = 70$ ,  $VaR_{95\%}[A] + VaR_{95\%}[B] = 140$ . Defaults are independent. Elementary calculations then establish that we receive a value at 140 with a probability of 0.16%, a value at 170 with a probability of  $4\% \times 96\% \times 2 = 7.68\%$ , and a value at 200 with a probability of 92.16%. Hence  $VaR_{95\%}[A + B] = 170$ . Thus,  $VaR_{95\%}[A] + VaR_{95\%}[B] < VaR_{95\%}[A + B]$ , and the VaR violates subadditivity.*

Final Event	A	B	A + B	Prob
1	70	70	140	$4\% \times 4\% = 0.16\%$
2	70	100	170	$4\% \times 96\% = 3.84\%$
3	100	70	170	$96\% \times 4\% = 3.84\%$
4	100	100	200	$96\% \times 96\% = 92.16\%$
$VaR_{95\%}$	70	70	170	

According to Heyde, et al. (2006) [31], VaR has been criticized because of its lack of subadditivity. However, VaR is subadditive for elliptically distributed,  $E_n(\mu, \Sigma, \phi)$  random vectors (defined in 2.3.1 below) as shown by see Embrechts (2002) [22].

**Theorem 2.7. Subadditivity of VaR.** *Suppose  $X_i \sim E_n(\mu, \Sigma, \phi)$  with  $\sigma^2[X_i] < \infty$  for all  $i$ . Let*

$$\mathcal{P} = \left\{ Z = \sum_{i=1}^n \lambda_i X_i \mid \lambda_i \in \mathbb{R} \right\}$$

*be the set of all linear portfolios. Then for any two portfolios  $Z_1, Z_2 \in \Gamma$  and  $\alpha \in [0.5, 1)$ ,*

$$Q_\alpha[Z_1 + Z_2] \leq Q_\alpha[Z_1] + Q_\alpha[Z_2].$$

*Proof.* Marginal distributions  $X_i$  for all  $i$  are elliptical so linear combinations of  $Z_1, Z_2, Z_1 +$

$Z_2$  have distributions of the same type. Let  $q_\alpha$  be the  $\alpha$ -quantile of the standardized distribution of this type. Then

$$\begin{aligned} Q_\alpha[Z_1] &= E[Z_1] + \sigma[Z_1]q_\alpha, \\ Q_\alpha[Z_2] &= E[Z_2] + \sigma[Z_2]q_\alpha, \\ Q_\alpha[Z_1 + Z_2] &= E[Z_1 + Z_2] + \sigma[Z_1 + Z_2]q_\alpha, \end{aligned}$$

Since  $\sigma[Z_1 + Z_2] \leq \sigma[Z_1] + \sigma[Z_2]$  and  $q_\alpha \geq 0$  the result follows.  $\square$   
Let us consider when the equality holds.

$$\sigma[Z_1 + Z_2] = \sigma[Z_1] + \sigma[Z_2] \text{ if and only if}$$

$$\sigma^2[Z_1] + \sigma^2[Z_2] + 2Cov[Z_1, Z_2] = \sigma^2[Z_1] + \sigma^2[Z_2] + 2\sigma[Z_1]\sigma[Z_2] \text{ if and only if}$$

$$Cov[Z_1, Z_2] = \sigma[Z_1]\sigma[Z_2].$$

That also means the correlation coefficient  $\rho = 1$ . It is reasonable that if  $Z_1, Z_2$  is a perfect positive linear relationship, then the VaR will satisfy additivity.

In the article I, there are some comments about VaR and additivity.

Although in the center of the distributions VaR may violate the subadditivity, Danielsson et al. (2005) [12] questioned whether the violation is merely a technical issue, at least if one focuses on the tail regions which are the most relevant regions for risk management. Indeed they showed that VaR is subadditive in the tail regions, provided that the tails in the joint distribution are not extremely fat (with tail index less than one). They also carried out simulations showing that VaR is indeed subadditive for most practical applications. Distributions with tail index less than one have very fat tails. They are difficult to find and easy to identify. Danielsson et al. (2005) [12] argued that they can be treated as special cases in financial modelling.

Uryasev (2010) [59] presents some pros and cons for VaR.

VaR: Pros

1. VaR is a relatively simple risk management concept and has a clear interpretation
2. Specifying VaR for all confidence levels completely defines the distribution

3. VaR focuses on the part of the distribution specified by the confidence level
4. Estimation procedures are stable
5. VaR can be estimated with parametric models

VaR: Cons

1. VaR does not account for properties of the distribution beyond the confidence level
2. Risk control using VaR may lead to undesirable results for skewed distributions
3. VaR is a non-convex and discontinuous function for discrete distributions

### 2.1.2.2 Tail Value at Risk

Following Dhaene et al. (2004) [18], a single quantile risk measure of a predetermined level  $\alpha$  assesses the ‘worst case’ loss, where worst case is defined as the event with a  $(1 - \alpha)$  probability. One problem with the quantile risk measure is that it does not take into consideration what the loss will be if that  $(1 - \alpha)$  worst case event actually occurs and does not give any information about the thickness of the upper tail of the distribution function from  $Q_\alpha[X]$  on. The loss distribution above the quantile does not affect the risk measure. A regulator is not only concerned with the frequency of default, but also about the severity of default. Also shareholders and management should be concerned with the question ‘how bad is bad?’ when they want to evaluate the risks at hand in a good way. Therefore, we also use another risk measure which is called the Tail Value at Risk (TVaR) at level  $\alpha$ . It is denoted by  $TVaR_\alpha[X]$  and defined by

$$TVaR_\alpha[X] = \frac{1}{1 - \alpha} \int_\alpha^1 Q_q[X] dq, \quad \alpha \in (0, 1). \quad (2.3)$$

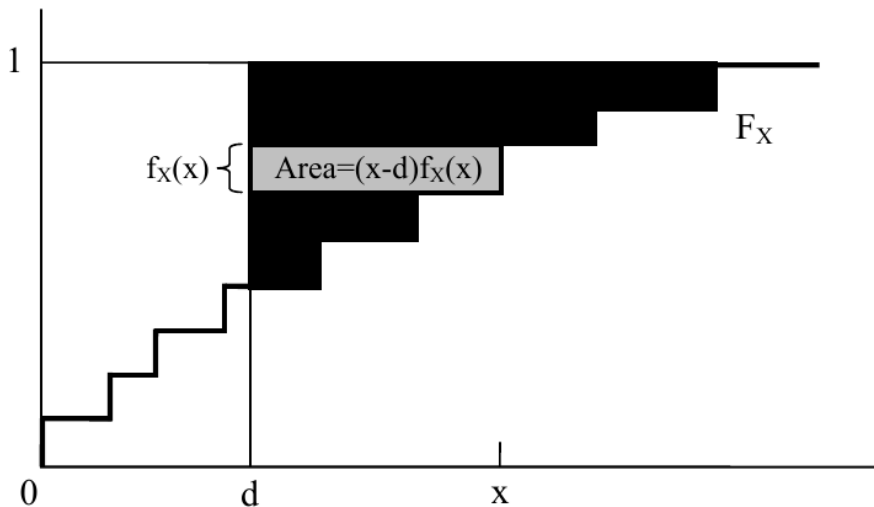
It is the arithmetic average of the quantiles of  $X$ , from  $\alpha$  on. Note that the TVaR is always larger than the corresponding quantile. From the equation (2.3) definition it follows immediately that the Tail Value at Risk is a non-decreasing function of  $\alpha$ .

Let  $X$  again denote the aggregate claims of an insurance and investment portfolio over a given reference period. We could define ‘bad times’ as those where  $X$  takes a value in the interval  $[Q_\alpha[X], TVaR_\alpha[X]]$ . Hence, ‘bad times’ are those where the aggregate claims exceed the threshold  $Q_\alpha[X]$ , but not using up all the available capital. The width of the interval is a ‘cushion’ that is used in case of ‘bad times’. For more details, see Overbeck (2000) [47].

The *Expected Shortfall (ESF)* at level  $\alpha$  will be denoted by  $ESF_\alpha[X]$ , and is defined as

$$ESF_\alpha[X] = E[(X - Q_\alpha[X])_+], \quad \alpha \in (0.5, 1). \quad (2.4)$$

This risk measure can be interpreted as the expected value of the *shortfall* of  $X$  and the quantile  $Q_\alpha[X]$ ,  $(X - Q_\alpha[X])_+ := \max(X - Q_\alpha[X], 0)$ .




---

FIGURE 2.1: Graphical derivation of stop-loss  $E[(X - d)_+]$  for a discrete cumulative distribution function from Karniychuk (2006) [36].

The following relation holds between these risk measures defined above.

**Theorem 2.8. Relation between Quantiles, TVaR and ESF.** For  $\alpha \in (0, 1)$ , we have that

$$TVaR_\alpha[X] = Q_\alpha[X] + \frac{1}{1-\alpha}ESF_\alpha[X]. \quad (2.5)$$

*Proof.* Expression (2.5) follows from

$$ESF_\alpha[X] = \int_0^1 (Q_q[X] - Q_\alpha[X])_+ dq = \int_\alpha^1 Q_q[X] dq - (1-\alpha)Q_\alpha[X].$$

□

Uryasev (2010) [59] presents some pros and cons for TVaR and the some observations for VaR and TVaR.

TVaR: Pros

1. TVaR has a clear engineering interpretation
2. Specifying TVaR for all confidence levels completely defines the distribution
3. TVaR is a coherent risk measure
4. TVaR is continuous.
5. TVaR is a convex function.

TVaR: Cons

1. TVaR is more sensitive than VaR to estimation errors.
2. TVaR accuracy is heavily affected by accuracy of tail modelling

Some observations for VaR and TVaR.

1. VaR does not control scenarios exceeding VaR and may lead to bearing high uncontrollable risk



2. VaR estimates are statistically more stable than TVaR estimates
3. TVaR is more sensitive than VaR to estimation errors
4. TVaR accuracy is heavily affected by accuracy of tail modelling

## 2.2 Dependence

If risks are independent, the calculation of risk measure is simple,. Mathematically it is easy to deal with independence. However in most real life situations we are not confronted with independent risks. Usually the risks from different business lines are dependent. For instance, business cycles affect different business lines in some way.

### 2.2.1 Correlation

#### 2.2.1.1 Pearson's correlation

The most familiar measure of dependence between two quantities is the Pearson product-moment correlation coefficient, or "Pearson's correlation." It is obtained by dividing the covariance of the two variables by the product of their standard deviations.

The population correlation coefficient  $\rho_{X,Y}$  between two random variables X and Y with expected values  $\mu_X$  and  $\mu_Y$  and standard deviations  $\sigma_X$  and  $\sigma_Y$  is defined as:

$$\rho[X, Y] = Cor(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y}, \quad (2.6)$$

where  $E$  is the expected value operator, *cov* means covariance, and, *cor* a widely used alternative notation for Pearson's correlation.

### 2.2.1.2 Rank correlation

Rank correlation is an alternative to the use of Pearson correlation as a measure of dependence. The two common types of rank correlation  $\rho_{rank}$  are:

1. Spearman coefficient; and
2. Kendall Tau correlation.

**Definition 2.2.** *The Spearman coefficient is defined as the Pearson correlation coefficient between the ranked variables.*

If we are given two vectors  $X = (X_1, \dots, X_n)$  and  $Y = (Y_1, \dots, Y_n)$  that present observations of the random variables  $X$  and  $Y$ , then Spearman coefficient  $\rho_S$  between  $X$  and  $Y$  is a Pearson correlation between the vectors of ranks of  $X_i$  and  $Y_i$ .

**Definition 2.3.** *Let  $(X_1, Y_1)$  and  $(X_2, Y_2)$  be independent and identically distributed random vectors. Then the population version of Kendall's tau is defined as:*

$$\tau = \tau_{X,Y} = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

The following property holds for rank correlation:

$$\rho_{rank}[T(X), T(Y)] = \rho_{rank}[X, Y] \tag{2.7}$$

for any non-linear monotonic transformation  $T$ .

## 2.2.2 Copula

In recent years, it is more and more popular to describe dependence by copula. Shaw et al. (2010) [49] comment that the copula approach is different from comparing with the covariance matrix. It involves a Monte Carlo simulation with the full marginal risk distribution of each risk and a copula function to produce a meaningful aggregate risk distribution. The copula is a convenient method for combining individual distributions into a multivariate distribution.

### 2.2.2.1 Introduction

Shaw et al. (2010) [49] give a very good introduction to the copula theory. Copulas are very flexible in that one can combine a varied number of marginal risk distributions together with a varying number of copula distributions.

In a simple case of two risks  $X_1$  and  $X_2$ , a copula  $C(u, v)$  is part of a mathematical expression of their joint distribution function  $F(x_1, x_2)$  in terms of the individual marginal risk distributions  $F_1(x_1)$  and  $F_2(x_2)$ :

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)). \quad (2.8)$$

Sklar's Theorem provides the theoretical foundation for the application of copulas.

**Theorem 2.9.** *Sklar's theorem*

*If  $F(x_1, \dots, x_n)$  is a joint distribution function with marginal distributions  $F_1(x_1), \dots, F_n(x_n)$ , then there exists a copula  $C$  such that*

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

### 2.2.2.2 Tail dependence

The following approach, as provided in the monograph of Joe (1997) [33], represents one of many possible definitions of tail dependence, see Schmidt [50].

Let  $X = (X_1, X_2)^T$  be a two-dimensional random vector. We say that  $X$  is (bivariate) upper tail-dependent if:

$$\lambda_U = \lim_{v \rightarrow 1^-} P\{X_1 > F_1^{-1}(v) | X_2 > F_2^{-1}(v)\} > 0, \quad (2.9)$$

in case the limit exists.  $F_1^{-1}$  and  $F_2^{-1}$  denote the generalized inverse distribution functions of  $X_1$  and  $X_2$ , respectively. Consequently, we say  $X = (X_1, X_2)^T$  is upper tail-independent if  $\lambda_U$  equals 0. Further, we call  $\lambda_U$  the upper tail-dependence coefficient (upper TDC).

In case  $X = (X_1, X_2)^T$  is standard normally or  $t$ -distributed, formula (2.9) simplifies to:

$$\lambda_U = \lim_{v \rightarrow 1^-} \lambda_U(v) = \lim_{v \rightarrow 1^-} 2P\{X_1 > F_1^{-1}(v) | X_2 = F_2^{-1}(v)\}. \quad (2.10)$$

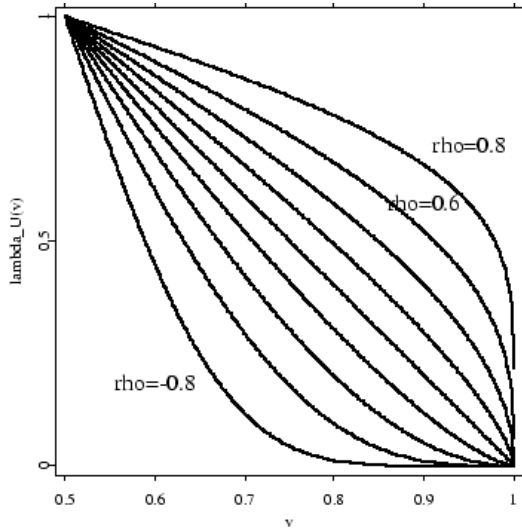


FIGURE 2.2: The function  $\lambda_U(v) = \lim_{v \rightarrow 1^-} 2P\{X_1 > F_1^{-1}(v) | X_2 = F_2^{-1}(v)\}$  for a bivariate normal distribution with correlation coefficients  $\rho = -0.8, -0.6, \dots, 0.6, 0.8$ . Note that  $\lambda_U = 0$  for all  $\rho \in (-1, 1)$ . The normal distribution fails to catch the tail dependence. The figure is from Schmidt [50].

Figures 2.2 and 2.3 illustrate tail dependence for a bivariate normal and  $t$ - distribution. Irrespectively of the correlation coefficient  $\rho$ , the bivariate normal distribution is (upper) tail independent. In contrast, the bivariate  $t$ -distribution exhibits (upper) tail dependence and the degree of tail dependence is affected by the correlation coefficient  $\rho$ .

**Example 2.10.** *Normal distribution*

*If  $X$  and  $Y$  are jointly normal and uncorrelated, then they are independent. The requirement that  $X$  and  $Y$  should be jointly normal is essential, without it the property does not hold. For non-normal random variables uncorrelated does not imply independence.*

*For the case  $\text{corr}[\mathbf{X}] \neq \pm 1$ , the normal distribution  $\mathbf{X}$  is tail-independent. Thus, the upper*

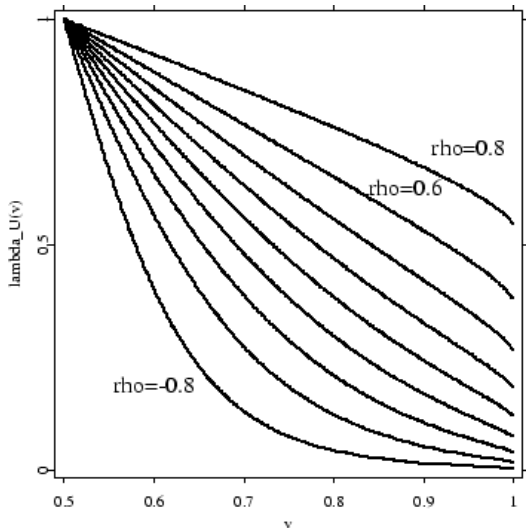


FIGURE 2.3: The function  $\lambda_U(v) = \lim_{v \rightarrow 1^-} 2P\{X_1 > F_1^{-1}(v) | X_2 = F_2^{-1}(v)\}$  for a bivariate  $t$ -distribution with correlation coefficients  $\rho = -0.8, -0.6, \dots, 0.6, 0.8$ . The figure is from Schmidt [50].

tail covariance  $TC_p[\mathbf{X}] = 0$ .

The Pearson correlation is  $+1$  in the case of a perfect positive (increasing) linear relationship (correlation),  $-1$  in the case of a perfect decreasing (negative) linear relationship (anticorrelation), and some value between  $-1$  and  $1$  in all other cases, indicating the degree of linear dependence between the variables.

Assume that  $\sigma_1 = \sigma_2 = 1$ . If correlation is  $1$ , we can think  $X_2 = X_1 + b$ , the upper tail covariance

$$TC_p[\mathbf{X}] = Cov[X_1, X_1 + b | X_1 > F_{X_1}^{-1}(p), X_2 > F_{X_2}^{-1}(p)] = 1, \quad p \in (0, 1). \quad (2.11)$$

### 2.2.2.3 FGM copula

A dependence structure for  $(X_1, X_2)$  based on the FGM copula is introduced.

**Theorem 2.11.** *Sklar's theorem.* For any bivariate distribution function  $H(x, y)$ , let  $F(x) = H(x, \infty)$  and  $G(y) = H(\infty, y)$  be the univariate marginal probability distribution

functions. Then there exists a copula  $C$  such that

$$H(x, y) = C(F(x), G(y)).$$

The copulas of the Farlie-Gumbel-Morgenstern family are defined by

$$C_\theta(u_1, u_2) = u_1 u_2 (1 + \theta(1 - u_1)(1 - u_2))$$

for  $u_i \in [0, 1]$ ,  $i = 1, 2$ , and dependence parameter  $\theta \in [-1, 1]$ . We simulated 500 observations from the two extreme members ( $\theta = -1$  and  $\theta = 1$ ) of this family using the R package `copula`, see Figure 2.4.

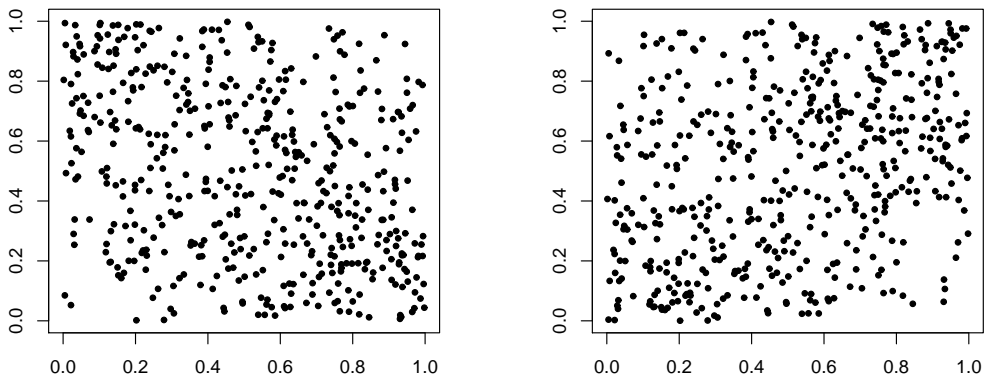


FIGURE 2.4: Scatter-plots for FGM copulas

**Definition 2.4.** Let  $(X_1, Y_1)$  and  $(X_2, Y_2)$  be independent and identically distributed random vectors. Then the population version of Kendall's tau is defined as:

$$\tau = \tau_{X,Y} = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

For FGM copulas,  $\tau_\theta = 2\theta/9 \in [-2/9, 2/9]$ . The details can be found in Nelsen (2006) [45], p. 162.

**Definition 2.5.** Let  $(X_1, Y_1)$ ,  $(X_2, Y_2)$  and  $(X_3, Y_3)$  be three independent and identically distributed random vectors. Then the population version of Spearman's rho is defined as:

$$\rho_{X,Y} = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]).$$

For FGM copulas,  $\rho_\theta = \theta/3 \in [-1/3, 1/3]$ . The details can be found in Nelsen (2006) [45], p. 168.

**Proposition 2.12.** Let  $X_1$  and  $X_2$  be two exponentially distributed with  $\lambda_1$  and  $\lambda_2$ , respectively. Suppose that the dependence is defined by the FGM copula with parameter  $\theta$ . Then the correlation between  $X_1$  and  $X_2$  is  $\rho_{X_1, X_2} = \theta/4 \in [-1/4, 1/4]$ .

**Proof.** Pearson's correlation is

$$\rho_{X_1, X_2} = \frac{Cov[X_1, X_2]}{\sqrt{Var[X_1]Var[X_2]}} = \frac{E[X_1X_2] - E[X_1]E[X_2]}{\sqrt{Var[X_1]Var[X_2]}}. \quad (2.12)$$

To calculate  $E[X_1X_2]$ , we need know that the joint cdf of  $(X_1, X_2)$  is

$$F_{X_1, X_2}(x_1, x_2) = H(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)(1 + \theta(1 - F_{X_1}(x_1))(1 - F_{X_2}(x_2)))$$

and the joint pdf

$$\begin{aligned} f_{X_1, X_2}(x_1, x_2) &= \frac{\partial^2 F_{X_1, X_2}(x_1, x_2)}{\partial x \partial y} = \frac{\partial^2 C_\theta(u_1, u_2)}{\partial u_1 \partial u_2} \frac{\partial u_1 \partial u_2}{\partial x \partial y} \\ &= \frac{\partial^2 C_\theta(u_1, u_2)}{\partial u_1 \partial u_2} f_{X_1}(x_1) f_{X_2}(x_2) \\ &= (1 + \theta(1 - 2u_1)(1 - 2u_2)) f_{X_1}(x_1) f_{X_2}(x_2) \\ &= (1 + \theta(1 - 2F_{X_1}(x_1))(1 - 2F_{X_2}(x_2))) f_{X_1}(x_1) f_{X_2}(x_2) \\ &= (1 + \theta(1 - 2(1 - e^{-\lambda_1 x_1}))(1 - 2(1 - e^{-\lambda_2 x_2}))) \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} \\ &= (1 + \theta) \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} + \theta 2 \lambda_1 e^{-2\lambda_1 x_1} 2 \lambda_2 e^{-2\lambda_2 x_2} \\ &\quad - \theta(2 \lambda_1 e^{-2\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} + \lambda_1 e^{-\lambda_1 x_1} 2 \lambda_2 e^{-2\lambda_2 x_2}) \end{aligned}$$

Thus, the expectation of  $X_1X_2$  is

$$\begin{aligned} E[X_1X_2] &= \int_0^{+\infty} \int_0^{+\infty} x_1x_2f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\ &= \int_0^{+\infty} \int_0^{+\infty} x_1x_2((1+\theta)\lambda_1e^{-\lambda_1x_1}\lambda_2e^{-\lambda_2x_2} + \theta 2\lambda_1e^{-2\lambda_1x_1}2\lambda_2e^{-2\lambda_2x_2} \\ &\quad - \theta(2\lambda_1e^{-2\lambda_1x_1}\lambda_2e^{-\lambda_2x_2} + \lambda_1e^{-\lambda_1x_1}2\lambda_2e^{-2\lambda_2x_2}))dx_1dx_2. \end{aligned}$$

Since

$$\int_0^{+\infty} \int_0^{+\infty} x_1x_2\lambda_1e^{-\lambda_1x_1}\lambda_2e^{-\lambda_2x_2}dx_1dx_2 = \int_0^{+\infty} x_1\lambda_1e^{-\lambda_1x_1}dx_1 \int_0^{+\infty} x_2\lambda_2e^{-\lambda_2x_2}dx_2 \quad (2.13)$$

and we know

$$\int_0^{+\infty} x\lambda e^{-\lambda x}dx = \frac{1}{\lambda},$$

then

$$\int_0^{+\infty} \int_0^{+\infty} x_1x_2\lambda_1e^{-\lambda_1x_1}\lambda_2e^{-\lambda_2x_2}dx_1dx_2 = \frac{1}{\lambda_1\lambda_2}.$$

So

$$\begin{aligned} E[X_1X_2] &= \frac{1+\theta}{\lambda_1\lambda_2} + \frac{\theta}{2\lambda_12\lambda_2} - \theta \left( \frac{1}{2\lambda_1\lambda_2} + \frac{1}{\lambda_12\lambda_2} \right) \\ &= \frac{1+\theta}{\lambda_1\lambda_2} + \frac{\theta}{4\lambda_1\lambda_2} - \frac{\theta}{\lambda_1\lambda_2} = \frac{4+\theta}{4\lambda_1\lambda_2}. \end{aligned}$$

From equation (2.12), the correlation

$$\rho_{X_1,X_2} = \frac{E[X_1X_2] - E[X_1]E[X_2]}{\sqrt{Var[X_1]Var[X_2]}} = \frac{\frac{4+\theta}{4\lambda_1\lambda_2} - \frac{1}{\lambda_1\lambda_2}}{\frac{1}{\lambda_1\lambda_2}} = \frac{\theta}{4}.$$

Thus  $\rho_{X_1,X_2} \in [-1/4, 1/4]$ . □

Even if the FGM copula introduces only weak dependence, it can include positive as well as negative dependence and independence by choosing different  $\theta$ . It is also known that the FGM copula is a Taylor approximation of order one of the Frank copula (see Nelsen (2006) [45], page 133), Ali-Mikhail-Haq copula and Plackett copula (see Nelsen (2006) [45], page 100).



### 2.2.3 Right Joint Excess Probability

In modelling a loss, there is usually considerable concern about the chance and sizes of large claims - in particular, the study of the (right) tail, see Boland (2007) [8]. In Shaw et al. (2010) [49], there is a description of the Right Joint Excess Probability (RJEP).

**Definition 2.6.** *For a pair of risks, the Right Joint Excess Probability is the joint probability that two risks are greater than some deemed threshold.*

$$RJEP(p) = P[F_X(x) > p, F_Y(y) > p] = P[x > F_X^{-1}(p), y > F_Y^{-1}(p)]. \quad (2.14)$$

For independent random variables, the value of  $RJEP(p)$  is  $(1 - p)^2$ .

## 2.3 Distributions

In this section, we present some of the classic distributions used to model losses in insurance and finance. Some of these distributions such as the exponential and gamma are frequently used in survival analysis and engineering application, see Boland (2007) [8]. We will also consider distributions such as Pareto and lognormal which are particularly appropriate for studying losses.

Here we are interested in heavy-tailed distributions from theoretical point of view. The empirical point of view was studied by Kaasik (2009) [34]. When dealing with some empirical data, which seem to be heavy-tailed, Kaasik (2009) [34] investigated how to find the right distribution with suitable parameters.

### 2.3.1 Elliptical and Log-Elliptical distributions

According to Valdez (2005) [64], the class of elliptical loss distribution models provides a generalization of the class of normal loss models. The class of elliptical distributions has been introduced in the statistical literature by Kelker (1970) [37] and widely discussed in

Fang et al. (1990) [23], see also Landsman and Valdez (2003) [41], Valdez and Dhaene (2003) [62], and Valdez and Chernih (2003) [61] for applications in insurance and actuarial science. Embrechts et al. (2002) [22] also provides a fair amount of discussion of this important class as a tool for modelling risk dependencies.

**Definition 2.7.** *The random vector  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)^T$  is said to have an **elliptical** distribution, written as  $\mathbf{Y} \sim \mathbf{E}_n(\boldsymbol{\mu}, \Sigma, \psi)$ , if its characteristic function can be expressed as*

$$\varphi_{\mathbf{Y}}(\mathbf{t}) = E[\exp(it^T \mathbf{Y})] = \exp(it^T \boldsymbol{\mu}) \psi\left(\frac{1}{2} \mathbf{t}^T \Sigma \mathbf{t}\right), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n), \quad (2.15)$$

for some  $n$ -dimensional column-vector  $\boldsymbol{\mu}$ , some  $n \times n$  semi positive-definite matrix  $\Sigma$  and scalar function  $\psi(\mathbf{t})$ , which is called the characteristic generator.

An elliptical distributed random vector  $\mathbf{Y} \sim \mathbf{E}_n(\boldsymbol{\mu}, \Sigma, \psi)$  does not necessarily have a probability density. A necessary condition for  $\mathbf{Y}$  to possess a density is that  $\text{rank}(\Sigma) = n$ . If  $\mathbf{Y}$  has a density  $f_{\mathbf{Y}}(\mathbf{y})$ , then it has the following form:

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{c_n}{\sqrt{|\Sigma|}} g_n \left[ \frac{1}{2} (\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{y} - \boldsymbol{\mu}) \right], \quad (2.16)$$

for some non-negative function  $g_n(\cdot)$ , which is called the *density generator*. The condition

$$\int_0^{\infty} y^{n/2-1} g_n(y) dy < \infty \quad (2.17)$$

guarantees that  $g_n(\cdot)$  is a density generator, see Fang et al. (1990) [23].

The normalizing constant  $c_n$  in (2.16) is given by

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^{\infty} y^{n/2-1} g_n(y) dy \right]^{-1}, \quad (2.18)$$

which is assumed to be finite. More details on the elliptical family of distributions can be found in [23], [41], amongst others.

**Example 2.13.** *The  $n$ -dimensional random vector  $\mathbf{Y}$  has the **multivariate normal** distribution with parameters  $\boldsymbol{\mu}$  and  $\Sigma$ , notation  $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \Sigma)$ , if its characteristic function*

is given by

$$E[\exp(it^T \mathbf{Y})] = \exp(it^T \boldsymbol{\mu}) \exp\left(-\frac{1}{2} \mathbf{t}^T \Sigma \mathbf{t}\right). \quad (2.19)$$

From (2.15) we see that  $N_n(\boldsymbol{\mu}, \Sigma)$  has an elliptical distribution with characteristic generator  $\psi$  given by

$$\psi(t) = \exp(-t). \quad (2.20)$$

Since  $\psi'(0) = -1$  the matrix  $\Sigma$  in (2.19) is the covariance matrix of  $\mathbf{Y}$ . In case  $\Sigma$  is positive definite, the random vector  $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \Sigma)$  has a density which is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{n/2} \sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu})\right]. \quad (2.21)$$

Comparing with (2.16), we find that the density generator  $g_n$  and the normalising constant  $c_n$  of  $N_n(\boldsymbol{\mu}, \Sigma)$  are given by

$$g_n(y) = \exp(-y) \quad \text{and} \quad c_n = \frac{1}{(2\pi)^{n/2}} \quad (2.22)$$

respectively.

**Definition 2.8.** The random vector  $\mathbf{X} = (X_1, X_2, \dots, X_n)^T$  is said to have a **log-elliptical** distribution, written as  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \Sigma, \psi)$ , if  $\mathbf{Y} = \ln \mathbf{X} = (\ln X_1, \ln X_2, \dots, \ln X_n)^T \sim \mathbf{E}_n(\boldsymbol{\mu}, \Sigma, \psi)$  with expectations  $\boldsymbol{\mu}$ , generalized covariance matrix  $\Sigma$  and characteristic generator  $\psi$ .

If the density of  $\mathbf{Y} = \ln \mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$  exists, then the density of  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \Sigma, \psi)$  also exists. From (2.16), it follows that the density of  $\mathbf{X}$  is equal to

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} \left( \prod_{k=1}^n x_k^{-1} \right) g_n \left[ \frac{1}{2} (\ln \mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\ln \mathbf{x} - \boldsymbol{\mu}) \right], \quad (2.23)$$

see Fang et al. (1990).

### 2.3.2 Exponential distribution

In Boland (2007) [8], the exponential distribution is one of the simplest and most basic distributions used in modelling. The random variable  $X$  is exponential distributed with parameter  $\lambda$ . Its density function is

$$f_X(x) = \lambda e^{-\lambda x} \quad (2.24)$$

and the survival function is

$$\bar{F}_X(x) = e^{-\lambda x} \quad (2.25)$$

for  $x > 0$ .

An exponential random variable  $X$  has the memoryless property in that for any  $M, x > 0$ ,

$$P(X > M + x | X > M) = P(X > x). \quad (2.26)$$

For an exponential distribution  $X$ , the tail probability  $\bar{F}_X(x) = P(X > x) = e^{-\lambda x}$  converge to 0 exponentially fast. In many situations, it may be appropriate to try and model a slower vanishing tail distribution. For example, if  $P(X > x)$  is of the form  $a^\alpha / (bx + c)^\alpha$  for certain positive constant  $a, b, c$  and  $\alpha$ , then the tail probability of  $X$  goes to 0 at a slower polynomial rate. For a function of the form  $a^\alpha / (bx + c)^\alpha$  to be the survival function of a positive random variable, one must have that  $p(X > 0) = (a/c)^\alpha = 1$ . This gives rise to the Pareto family of distributions.

### 2.3.3 Pareto distribution

We use the introduction of Pareto distribution from in Boland (2007) [8].

The random variable  $X$  is Pareto with positive parameters  $a$  and  $b$  if it has density function

$$f(x) = \frac{ab^a}{(x+b)^{a+1}} \quad (2.27)$$

or equivalently, survival function

$$\bar{F}_X(x) = \left( \frac{b}{x+b} \right)^a \quad (2.28)$$

for  $x > 0$ . The Pareto distribution is named after Vilfredo Pareto (1848-1923) who used it in modelling welfare economics. Today it is commonly used to model income distribution in economics or claim-size distribution in insurance.

Like the exponential family of random variables, the Pareto distributions have density and survival function which are very tractable. Pareto random variables have some nice preservation properties. For example, if  $X \sim \text{Pareto}(a, b)$  and  $k > 0$ , then  $kX \sim \text{Pareto}(a, kb)$  since

$$P(kX > x) = P(X > x/k) = \left( \frac{b}{x/k+b} \right)^a = \left( \frac{kb}{x+kb} \right)^a. \quad (2.29)$$

This property is useful in dealing with inflation in claims. Moreover, if  $M > 0$ , then

$$P(X > M+x | X > M) = \left( \frac{b}{x+M+b} \right)^a / \left( \frac{b}{M+b} \right)^a = \left( \frac{M+b}{x+M+b} \right)^a, \quad (2.30)$$

which implies that if  $X > M$ , then  $X - M$  (or the excess of  $X$  over  $M$ ) is  $\text{Pareto}(a, M+b)$ . The property is useful in evaluating the effect of deductibles and/or excess levels for insurance in handling losses.

### 2.3.4 Gamma distribution

The gamma family of probability distribution is both versatile and useful, in Boland (2007) [8]. The gamma function is defined for any  $\alpha > 0$  by

$$\Gamma(\alpha) = \int_0^{+\infty} y^{\alpha-1} e^{-y} dy, \quad (2.31)$$

and has the properties that

$$\Gamma(n) = (n-1)\Gamma(n-1) \quad (2.32)$$

and  $\Gamma(1/2) = \sqrt{\pi}$ .

The gamma distribution  $X$  can be parametrized in terms of a shape parameter  $\alpha$  and an inverse scale parameter  $\beta$ , called a rate parameter:

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad (2.33)$$

for  $x \geq 0$  and  $\alpha, \beta > 0$ .

If  $X \sim \Gamma(\alpha, \lambda)$ , then Moment-generating function (mgf)  $M_X(t) = [\beta/(\beta - t)]^\alpha$  for  $t < \beta$ ,  $E[X] = \alpha/\beta$  and  $Var[X] = \alpha/\beta^2$ . In a Poisson process where events are occurring at the rate of  $\beta$  per unit time, the waiting time  $T_\alpha$  until the  $\alpha$ -th event "arrives" has a gamma distribution with parameters  $\alpha$  and  $\beta$ ,  $T_\alpha \sim \Gamma(\alpha, \beta)$ .

when the shape parameter  $\alpha = 1$ , we obtain the exponential distributions. Moreover, the  $\Gamma(v/2, 1/2)$  distributions is the  $\chi^2$  distribution with  $v$  degrees of freedom. Hence the gamma family includes both the exponential and  $\chi^2$  distributions.



# Chapter 3

## Capital allocation

### 3.1 How to allocate?

Consider a portfolio of  $n$  individual losses as  $\mathbf{X} = (X_1, X_2, \dots, X_n)^T$  at a fixed date. Denote each economic capital for loss  $X_i$  by  $\rho[X_i]$ .

The aggregate loss is defined by the sum

$$S = \sum_{i=1}^n X_i, \quad (3.1)$$

and the total economic capital  $K = \rho[S]$ .

The allocation problem is to determine a capital  $K_i$  for each loss  $X_i$  and

$$\sum_{i=1}^n K_i = K, \quad K_i \geq 0. \quad (3.2)$$

Usually, the total economic capital is smaller than the sum of the economic capital for each risk,  $K < \sum_{i=1}^n \rho[X_i]$  because there is diversification benefit. Allocating the total capital back to the lower levels also means allocating the diversification benefit to individual risks.



## 3.2 Different principles

The section will present different principles of capital allocation and illustrates with the example from Corrigan et al. (2009) [10].

There are three portfolios of company  $\mathbf{X} = (X_1, X_2, X_3)^T$ . Losses from portfolios are assumed to be normally distributed, with mean  $\boldsymbol{\mu} = (50, 40, 70)^T$  and standard deviation  $\boldsymbol{\sigma} = (10, 7, 12)^T$ .

The following correlation are specified between the 3 portfolios:

$$Corr = \begin{pmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.2 \\ 0.3 & 0.2 & 1 \end{pmatrix} \quad (3.3)$$

$VaR_{0.995}$  is used as a risk measure to determine the capital requirement for each of  $X_1, X_2$  and  $X_3$  in this example; in practice, any risk measure can be used. The risk measures are :

$$\rho[\mathbf{X}] = VaR_{0.995}[\mathbf{X}] = (75.9, 58.8, 102.7)^T. \quad (3.4)$$

We sum up the risk  $S = X_1 + X_2 + X_3$  and the total risk is a normal distribution with mean  $\mu_S = 50 + 40 + 70 = 160$  and stand deviation

$$\sigma_S = \sqrt{\boldsymbol{\sigma}[\mathbf{X}]^T Corr \boldsymbol{\sigma}[\mathbf{X}]} = \sqrt{(10, 7, 12) \begin{pmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.2 \\ 0.3 & 0.2 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 7 \\ 12 \end{pmatrix}} = 22.6. \quad (3.5)$$

The total economic capital is  $K = \rho[S] = VaR_{0.995}[S] = 227.0$ .

The results presented in the table.

Note, the sum of parts  $\sum \rho[X_i] = 237.4$  is greater than the total portfolio capital requirement  $K = 227.0$ . The aim of this example is to allocate the  $K = 227.0$  back to portfolio  $X_1, X_2$  and  $X_3$ .

TABLE 3.1: Capital allocation

Portfolio	$\mu$	$\sigma$	$VaR_{0.995}$
$X_1$	50	10	75.9
$X_2$	40	7	58.8
$X_3$	70	12	102.7
Sum S	160	22.6	227.0

### 3.2.1 Proportional capital contribution

The total capital requirement is allocated linearly to each loss  $X_i$ . The capital for each loss can be calculated by:

$$K_i = \frac{K}{\sum_{i=1}^n \rho[X_i]} \rho[X_i], \quad i = 1, \dots, n. \quad (3.6)$$

In this example, each receive the allocations of  $28.2/44.7(15.8, 8.0, 20.9)^T = (9.9, 5.1, 13.2)^T$ , see the following table.

TABLE 3.2: Proportional capital allocation

Portfolio	Capital required $\rho[X_i]$	Proportional allocation
$X_1$	75.9	72.5
$X_2$	58.8	56.2
$X_3$	102.7	98.2
Sum	237.4	227.0
Capital requirement $K$ for $S$	227.0	

The advantage is very simple. The disadvantage is that diversification benefit allocated is in proportion to capital requirement but not correlation. Hence, the portfolio with higher mean obtains greater diversification benefit. However, the portfolio with lower correlation with the total portfolios should gain greater diversification benefit.

To improve the principle, we extend the risk measure  $\rho[X_i]$  to  $\rho[X_i, S]$ , including the sum  $S = \sum_{i=1}^n X_i$ .

### 3.2.2 Covariance allocation principle

Covariance allocation principle is a special case of proportional capital allocation when  $\rho[X_i, S] = Cov[X_i, S]$ .

According to Dhaene et al. (2011) [19], the covariance allocation principle proposed by e.g. Overbeck (2000) [47] is given by

$$K_i = \frac{Cov[X_i, S]}{Var[S]}K, \quad i = 1, \dots, n, \quad (3.7)$$

where  $Cov[X_i, S]$  is the covariance between the individual loss  $X_i$  and the aggregate loss  $S$  and  $Var[S]$  is the variance of  $S$ . Because clearly the sum of these individual covariances is equal to the variance of the aggregate loss,  $\sum Cov[X_i, S] = Var[S]$ , the full allocation requirement is automatically satisfied in this case,  $\sum K_i = K$ .

The covariance allocation rule explicitly takes into account the dependence structure of the random losses  $(X_1, \dots, X_n)$ . Business lines with a loss that is more correlated with the aggregate loss  $S$  are penalised by requiring them to hold a larger amount of capital than those which are less correlated.

The advantage of this principle is that it is easy to calculate and only need know the covariances  $Cov[X_i, X_j]$  between different risks. The allocation percentage  $Cov[X_i, S]/Var[S]$  to a line of business is the sum of the appropriate row of the covariance matrix  $Cov[X]$  divided by the sum of all elements, see the Covariance method from the website, Pricing Wiki.

In this example, the covariance matrix

$$Cov[X] = \sigma[\mathbf{X}] Corr \sigma[\mathbf{X}]^T = \begin{pmatrix} 10 \\ 7 \\ 12 \end{pmatrix} \begin{pmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.2 \\ 0.3 & 0.2 & 1 \end{pmatrix} (10, 7, 12) = \begin{pmatrix} 10 & 56 & 36 \\ 56 & 49 & 16.8 \\ 36 & 16.8 & 144 \end{pmatrix}.$$

The allocation percentage  $k_i = \frac{Cov[X_i, S]}{Var[S]}$  and allocation can be found in following Table.

Other allocation principles require more information, such as the joint distribution of the  $X$ 's. Even knowing the distribution, it is usually difficult to find an explicit formula and so we often need to calculate them from simulation data. The covariance allocation principle

TABLE 3.3: Covariance capital allocation

Portfolio	allocation percentage %	Covariance allocation
$X_1$	37.6	85.4
$X_2$	23.9	54.1
$X_3$	38.5	87.5
Sum		227.0

models dependence explicitly, which is an advantage.

The covariance allocation principle is, however, not a good way to allocate capital if the role of our risk capital is to cover claims from all lines of business.

Let us assume that there are two losses, constant  $C$  and random variable  $X$ . The sum of the two losses is  $S = C + X$ .

If the economic capital is  $K = VaR_q[S]$  and  $q \in (0, 1)$ , then the capital for this case is

$$K = VaR_q[S] = VaR_q[C + X] = C + VaR_q[X]. \quad (3.8)$$

According to the covariance allocation principle, the allocation to the constant loss is

$$K_1 = \frac{Cov[C, S]}{Var[S]} K = 0 \quad (3.9)$$

and for the other loss, allocation is all the capital  $K$ .

Non-risky loss still needs some share of the capital. Economic capital  $K$  should be sufficient to pay the liabilities of the company, both risky ones and non-risky ones.

It can be noted that the allocation percentages  $Cov[X_i, S]/Var[S]$  are not dependent on the level  $q$ . The allocation  $K_i$  can be a function of the level  $q$  if the total allocated capital  $K$  is the function of  $q$ , for instance,  $K = VaR_q[S]$ .

### 3.2.3 Discrete marginal contributions

Discrete marginal contribution of  $X_i$  = the capital requirement of the total portfolio  $K$  - the capital requirement of the total portfolio excluding the portfolio  $X_i$ , denoted by  $K_{\bar{i}}$ .

For example, when considering  $X_1$ , check the capital of portfolio  $K_{\bar{1}} = S_{\bar{1}} = X_2 + X_3$ .  $S_{\bar{1}}$  is a normal distribution with mean  $\mu_{S_{\bar{1}}} = 40 + 70 = 110$  and stand deviation

$$\sigma_{S_{\bar{1}}} = \sqrt{\sigma[\mathbf{X}_{23}]^T \text{Corr} \sigma[\mathbf{X}_{23}]} = \sqrt{(7, 12) \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 12 \end{pmatrix}} = 15.1. \quad (3.10)$$

Aggregate the capital requirements of  $X_2$  and  $X_3$ :

$$K_{\bar{1}} = VaR_{0.995}[S_{\bar{1}}] = 148.8. \quad (3.11)$$

The required capital of  $X_1X_3$  and  $X_1X_2$  are 165.8 and 124.3 separately.

In next table, the discrete marginal contributions of portfolio  $X_1, X_2$  and  $X_3$  are  $227.0 - 148.8 = 78.2$ ,  $227.0 - 165.8 = 61.2$  and  $227.0 - 124.3 = 102.7$  separately. Then scale them with the sum as the capital requirement of total portfolio 227.0. For example, the scaled marginal contribution of  $X_1$  is  $78.2 * 227.0 / 242.1 = 73.3$ .

TABLE 3.4: Discrete marginal capital allocation

Portfolio	Capital requirement excluding this one $K_{\bar{i}}$	Discrete marginal contribution $K - K_{\bar{i}}$	Scaled marginal contribution	Proportional allocation
$X_1$	148.8	78.2	73.7	72.5
$X_2$	165.8	61.2	57.3	56.2
$X_3$	124.3	102.7	96.3	98.2
<b>S</b>	227.0	242.1	227.0	227.0

Compared with proportional allocation, the capital requirement of portfolio  $X_1$  and  $X_3$  is more in scaled marginal contribution and  $X_2$  less. Comparably great positive correlation between  $X_1$  and  $X_2$  makes  $X_1$  more allocation in marginal principles.

**Note 2.** *If all the correlations between portfolios are equal to 1, discrete marginal contribution, scaled marginal contribution and proportional allocation will be the same.*

The advantage is that this is a more sophisticated approach. The disadvantage is that the calculation may generate negative values for capital requirements, for example, if there is one or more negative correlations between the portfolios.

### 3.3 Optimal capital allocations

According to Dhaene et al. (2011) [19], consider a portfolio of  $n$  individual losses  $X_1, X_2, \dots, X_n$  at a fixed future date  $T$ . Assume that  $\mathbf{X} = (X_1, X_2, \dots, X_n)^T$  is a random vector on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We will always assume that any loss  $X_i$  has a finite mean. The distribution function  $P[X_i \leq x]$  of  $X_i$  will be denoted by  $F_{X_i}(x)$ . We can look for a method of allocation from the optimal capital allocation problem.

Give the aggregate capital  $K > 0$ , allocate the capital  $K_i$  to business line  $i$  by solving

$$\min_{K_1, \dots, K_n} \sum_{j=1}^n \frac{1}{v_j} E[(X_j - K_j)^2 g(\mathbf{X})], \text{ such that } \sum_{j=1}^n K_j = K, \quad (3.12)$$

where the  $v_j$  are non-negative real numbers such that  $\sum_{j=1}^n v_j = 1$ , the  $g(\mathbf{X})$  are non-negative random variables such that  $E[g(\mathbf{X})] = 1$ .

Dhaene et al. (2011) [19] explain  $v_j$  as a measure of exposure or business volume of the  $j$ -th unit, such as revenue, insurance premium, etc. These scalar quantities are chosen such that they sum to 1. These  $v_j$  are used as weights attached to the different values of  $E[(X_j - K_j)^2 g(\mathbf{X})]$  in the minimization problem (3.12), in order to reflect the relative importance of the different business units. The non-negative function  $g(\mathbf{X})$  are used as the portfolio performance weight factor to the outcomes of the deviations  $(X_j - K_j)^2$ . The allocations based on  $g(\mathbf{X})$  will be called portfolio driven allocations.

**Theorem 3.1.** *The optimal allocation problem (3.12) has the following unique solution:*

$$K_i = E[X_i g(\mathbf{X})] + v_i \left( K - \sum_{j=1}^n E[X_j g(\mathbf{X})] \right), \quad i = 1, \dots, n, \quad (3.13)$$

where the  $v_j$  are non-negative real numbers such that  $\sum_{j=1}^n v_j = 1$ , the  $g(\mathbf{X})$  are non-negative random variables such that  $E[g(\mathbf{X})] = 1$ .

We define the volumes  $v_i$  by

$$v_i = \frac{E[X_i g(\mathbf{X})]}{\sum_{j=1}^n E[X_j g(\mathbf{X})]}, \quad i = 1, \dots, n,$$

we find that the capital allocation principle (3.13) reduces to the proportional allocation rule

$$K_i = \frac{E[X_i g(\mathbf{X})]}{\sum_{j=1}^n E[X_j g(\mathbf{X})]} K, \quad i = 1, \dots, n. \quad (3.14)$$

In Dhaene et al. (2011) [19], there are tables showing different  $E[X_j g(\mathbf{X})]$  by choosing different  $g(\mathbf{X})$ .

TABLE 3.5: Business lines driven risk measures

Risk measure	$g(X_i)$	$E[X_i g(X_i)]$
Standard deviation principle Bühlmann (1970) [9]	$1 + a \frac{X_i - E[X_i]}{\sigma_{X_i}}, a \geq 0$	$E[X_i] + a \sigma_{X_i}$
Tail VaR Overbeck (2000) [47]	$\frac{1}{1-q} \mathbf{1}(X_i > F_{X_i}^{-1}(q)), q \in (0, 1)$	$TVaR[X_i]$
Distortion risk measure Wang (1996) [71], Acerbi (2002) [1]	$h'(\bar{F}_{X_i}(X_i)), h : [0, 1] \rightarrow [0, 1],$ $h' > 0, h'' < 0$	$E[X_i h'(\bar{F}_{X_i}(X_i))]$
Exponential principle Gerber (1974) [27]	$\int_0^1 \frac{e^{\gamma a X_i}}{E[e^{\gamma a X_i}]} d\gamma, a > 0$	$\frac{1}{a} \ln E[e^{a X_i}]$
Esscher principle Gerber (1981) [28]	$\frac{e^{a X_i}}{E[e^{a X_i}]}, a > 0$	$\frac{E[X_i e^{a X_i}]}{E[e^{a X_i}]}$

In this thesis, we will use the proportional allocation

$$K_i = \frac{K}{\sum_{i=1}^n \rho[X_i, S]} \rho[X_i, S], \quad i = 1, \dots, n. \quad (3.15)$$

By choosing different risk measures  $\rho[X_i, S]$ , we can obtain different capital allocations.

TABLE 3.6: Aggregate portfolio driven allocations

Reference	$g(S)$	$E[X_i g(S)]$
Overbeck (2000) [47]	$1 + a \frac{S - E[S]}{\sigma_{X_i}}, a \geq 0$	$E[X_i] + a \frac{Cov[X_i, S]}{\sigma_S}$
Overbeck (2000) [47]	$\frac{1}{1-q} \mathbf{1}(S > F_S^{-1}(q)), q \in (0, 1)$	$E[X_i   S > F_S^{-1}(q)]$
Tsanakas (2004) [56]	$h'(\bar{F}_S(S)), h : [0, 1] \rightarrow [0, 1],$ $h' > 0, h'' < 0$	$E[X_i h'(\bar{F}_S(S))]$
Tsanakas (2009) [58]	$\int_0^1 \frac{e^{\gamma a S}}{E[e^{\gamma a S}]} d\gamma, a > 0$	$E[X_i \int_0^1 \frac{e^{\gamma a S}}{E[e^{\gamma a S}]} d\gamma]$
Wang (2007) [72]	$\frac{e^{aS}}{E[e^{aS}]}, a > 0$	$\frac{E[X_i e^{aS}]}{E[e^{aS}]}$





# Chapter 4

## Capital allocation based on tails

### 4.1 Conditional Expectation as Risk Measures

A risk measure is defined as a mapping  $\rho$  from a set  $\Gamma$  of random variables defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  to the real numbers  $\mathbb{R}$ :

$$\rho : \Gamma \rightarrow \mathbb{R} : \mathbf{X} \in \Gamma \rightarrow \rho[\mathbf{X}]$$

This set of random variables  $\mathbf{X}$  represents the losses associated with conducting a business. The conditional Expectation

$$CE[X_i] = E \left[ X_i | \cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\} \right], \quad i = 1, \dots, \dots, n, \quad p_j \in (0, 1)$$

can be a risk measure. The probability on the condition is also called Right Joint Excess Probability, see **Definition 2.6.**.

It satisfies that the properties for coherent risk measure, see 2.1.1. Hence, the Conditional Expectation  $CE[X_i]$  is a coherent risk measure.

Conditional Expectation can be a special case derived from optimal capital allocation problem, see **Theorem 3.1.**. A particular choice of the random variable  $g(\mathbf{X})$  considered

in (3.13) is given by

$$g(\mathbf{X}) = \frac{\mathbf{1}_{\cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}}}{P\left(\cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}\right)},$$

where  $\mathbf{1}_A$  is the indicator function of event  $A$ . Here the event  $\cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}$  is when each loss  $X_j$  exceeds some certain threshold, the quantile  $F_{X_j}^{-1}(p_j)$  for some  $p_j \in (0, 1)$ . We check

$$E[g(\mathbf{X})] = 1,$$

which satisfies the assumption. In this case, we find that

$$E[X_i g(\mathbf{X})] = E\left[X_i | \cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}\right], \quad i = 1, \dots, n. \quad (4.1)$$

The special case

$$E[X_i g(\mathbf{X})] = E[X_i | X_i > F_{X_i}^{-1}(p_i)] = CTE_{p_i}[X_i], \quad i = 1, \dots, n,$$

is called the Tail Value at Risk (TVaR) or sometimes Conditional Tail Expectation (CTE) of the loss  $X_i$ , see [47].

We can write the capital allocation (3.14) as follows

$$\begin{aligned} K_i &= \frac{E[X_i g(\mathbf{X})]}{\sum_{j=1}^n E[X_j g(\mathbf{X})]} K = \frac{E\left[X_i | \cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}\right]}{\sum_{j=1}^n E\left[X_j | \cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}\right]} K \\ &= \frac{E\left[X_i | \cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}\right]}{E\left[\sum_{j=1}^n X_j | \cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}\right]} K, \quad i = 1, \dots, n. \end{aligned} \quad (4.2)$$

TABLE 4.1: Conditional Expectation (CE) allocation

Risk measure	$g(\mathbf{X})$	$E[X_i g(\mathbf{X})]$
CE	$\frac{\mathbf{1}_{\cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}}}{P(\cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\})}, p_j \in (0, 1)$	$E\left[X_i   \cap_{j=1}^n \{X_j > F_{X_j}^{-1}(p_j)\}\right]$

The advantage is its simplicity. The disadvantage is not considering the tail variance. There are results of the Conditional Expectation with Pareto distributions in Wang (2012) [68].

## 4.2 Upper tail covariance

Consider the random variable (r.v.)  $X$  representing the claims related to an insurance portfolio over a given period. The cumulative distribution function (cdf) and the probability density function (pdf) of  $X$  are denoted by  $F_X(x)$  and  $f_X(x)$ , respectively. Suppose that the insurer is concerned about the claims related  $X$  exceeding a certain threshold, e.g. the quantile  $F_X^{-1}(p)$ , which is defined by

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \quad p \in (0, 1). \quad (4.3)$$

Valdez (2004) [63] defined the tail covariance between two random variables. For a bivariate vector  $\mathbf{X}^T = (X_1, X_2)$ , the tail covariance of  $\mathbf{X}^T$ , conditional on  $X_2 > F_{X_2}^{-1}(p)$  is

$$TCC_p[X_1|X_2] = Cov[X_1, X_2 | X_2 > F_{X_2}^{-1}(p)]. \quad (4.4)$$

This gives the information about the relationship of two claims when one exceeding some certain threshold.

We are also interested the relationship of two claims when both exceeding some certain thresholds. Here we use the Right Joint Excess Probability (see **Definition 2.6.**) as the condition. There is the definition of the Upper Tail Covariance as follows:

**Definition 4.1.** Consider a bivariate vector  $\mathbf{X} = (X_1, X_2)^T$  and a given probability level  $p$ , the (Upper) Tail Covariance  $TC_p[\mathbf{X}]$  of  $\mathbf{X}$ , conditional on  $X_1 > F_{X_1}^{-1}(p)$ ,  $X_2 > F_{X_2}^{-1}(p)$  is defined to be

$$TC_p[\mathbf{X}] = Cov[X_1, X_2 | X_1 > F_{X_1}^{-1}(p), X_2 > F_{X_2}^{-1}(p)], \quad p \in (0, 1). \quad (4.5)$$

**Note 3.** From Theorem 2 in [40], if  $\mathbf{X}$  is a comonotonic random vector and  $S$  is defined by  $S = X_1 + X_2$ , then

$$\text{Cov}[X_1, X_2 | X_1 > F_{X_1}^{-1}(p), X_2 > F_{X_2}^{-1}(p)] = \text{Cov}[X_1, X_2 | S > F_S^{-1}(p)]. \quad (4.6)$$

We can extend the idea to

$$TC_{(p_1, p_2)}[\mathbf{X}] = \text{Cov}[X_1, X_2 | X_1 > F_{X_1}^{-1}(p_1), X_2 > F_{X_2}^{-1}(p_2)], \quad p_1, p_2 \in (0, 1). \quad (4.7)$$

Here the probability levels  $p_1$  and  $p_2$  are not necessarily same.

Or when considering random variables  $\mathbf{X} = (X_1, X_2, \dots, X_n)^T$ , we can also set any conditions from  $X_1 > F_{X_1}^{-1}(p_1), X_2 > F_{X_2}^{-1}(p_2), \dots, X_n > F_{X_n}^{-1}(p_n)$ , for example considering

$$\text{Cov}[X_i, X_j | \cap_k X_k > F_{X_k}^{-1}(p_k)], \quad p_k \in (0, 1), \quad (4.8)$$

where  $i, j$  and  $k$  can be any natural numbers from 1 to  $n$ . This also makes Valde's tail covariance  $TCC$  a special case, see (4.4).

To keep the expression simple, in this thesis we investigate the Upper Tail Covariance from the **Definition 1**. Later we can see that it is trivial to derive the expression for (4.8).

Landsman et al. (2013) [40] derive expressions for the Upper Tail Variance of univariate log-elliptical distributions. In this thesis, we will consider upper tail covariance within the class of bivariate log-elliptical distributions.

## 4.2.1 The Upper Tail Covariance of bivariate log-elliptical distributions

Throughout this thesis we will only consider elliptical distributions which have a probability density, and hence have a continuous cumulative distribution function.

We will use the notations  $f_{\mathbf{X}}$ ,  $F_{\mathbf{X}}$  and  $\overline{F}_{\mathbf{X}}$  to denote the probability density function (pdf), the cumulative and the decumulative distribution function of the random vector  $\mathbf{X}$ .

**Theorem 4.1.** Let 2-dimensional random vector  $\mathbf{X} = (X_1, X_2)^T \sim LE_2(\boldsymbol{\mu}, \Sigma, \psi)$ , with  $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$  and the positive definite matrix  $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$ , then  $\mathbf{Y} = (Y_1, Y_2)^T \stackrel{d}{=} (\ln X_1 - \mu_1, \ln X_2 - \mu_2)^T \sim E_2(\mathbf{0}, \Sigma, \psi)$ . The upper tail covariance can be expressed as

$$TC_p[\mathbf{X}] = \frac{e^{\mu_1 + \mu_2} \psi\left(\frac{-\sigma_1^2 - 2\sigma_{12} - \sigma_2^2}{2}\right) \bar{F}_{\mathbf{Y}^*}(F_{Y_1}^{-1}(p), F_{Y_2}^{-1}(p))}{\bar{F}_{X_1, X_2}(F_{X_1}^{-1}(p), F_{X_2}^{-1}(p))} - \frac{e^{\mu_1 + \mu_2} \psi\left(\frac{-\sigma_1^2}{2}\right) \psi\left(\frac{-\sigma_2^2}{2}\right) \bar{F}_{\mathbf{Y}^{*1}}(F_{Y_1}^{-1}(p), F_{Y_2}^{-1}(p)) \bar{F}_{\mathbf{Y}^{*2}}(F_{Y_1}^{-1}(p), F_{Y_2}^{-1}(p))}{(\bar{F}_{X_1, X_2}(F_{X_1}^{-1}(p), F_{X_2}^{-1}(p)))^2}, \quad (4.9)$$

where  $p \in (0, 1)$ ,  $\mathbf{Y}^*$ ,  $\mathbf{Y}^{*1}$  and  $\mathbf{Y}^{*2}$  are r.v. with the probability densities given by

$$f_{\mathbf{Y}^*}(y_1, y_2) = \frac{e^{y_1 + y_2}}{\psi\left(\frac{-\sigma_1^2 - 2\sigma_{12} - \sigma_2^2}{2}\right)} f_{Y_1, Y_2}(y_1, y_2) \quad \text{and} \quad f_{\mathbf{Y}^{*k}}(y_1, y_2) = \frac{e^{y_k}}{\psi\left(\frac{-\sigma_k^2}{2}\right)} f_{Y_1, Y_2}(y_1, y_2) \quad (4.10)$$

when  $\psi\left(\frac{-\sigma_1^2 - 2\sigma_{12} - \sigma_2^2}{2}\right)$  and  $\psi\left(\frac{-\sigma_k^2}{2}\right)$  exist with  $k = 1, 2$ . The decumulative distribution functions

$$\bar{F}_{\mathbf{Y}^*}(F_{Y_1}^{-1}(p), F_{Y_2}^{-1}(p)) = \int_{F_{Y_2}^{-1}(p)}^{\infty} \int_{F_{Y_1}^{-1}(p)}^{\infty} f_{\mathbf{Y}^*}(y_1, y_2) dy_1 dy_2, \quad (4.11)$$

$$\bar{F}_{\mathbf{Y}^{*k}}(F_{Y_1}^{-1}(p), F_{Y_2}^{-1}(p)) = \int_{F_{Y_2}^{-1}(p)}^{\infty} \int_{F_{Y_1}^{-1}(p)}^{\infty} f_{\mathbf{Y}^{*k}}(y_1, y_2) dy_1 dy_2, \quad (4.12)$$

with  $k = 1, 2$  and

$$\bar{F}_{X_1, X_2}(F_{X_1}^{-1}(p), F_{X_2}^{-1}(p)) = \int_{F_{X_2}^{-1}(p)}^{\infty} \int_{F_{X_1}^{-1}(p)}^{\infty} f_{X_1, X_2}(x_1, x_2) dx_1 dx_2. \quad (4.13)$$

Proof and more discussions see Wang (2012) [69].

### 4.3 Definition of the Tail Covariance Premium Adjusted (TCPA) based allocation

The definition of the Tail Covariance Premium Adjusted (TCPA) is based on the Tail Covariance Premium (TCovP), a risk measure introduced by Furman and Landsman (2006) [25].

Consider risk  $X$  to be a random variable with cumulative distribution (cdf) and probability density function (pdf)  $F_X(x)$  and  $f_X(x)$ , respectively.

The Value at Risk (VaR) at level  $q$ ,  $0 < q < 1$ , of  $X$  is defined by

$$VaR_q(X) = \inf\{x_q : F_X(x_q) \geq q\}. \quad (4.14)$$

The Tail VaR (TVaR) is defined by

$$TVaR_q(X) = E[X | X > x_q]. \quad (4.15)$$

This is the expectation of the right tail. We are also interested in the dispersion along the right tail. Furman and Landsman (2006) [25] refer to this measure as Tail Variance (TV), and it is the conditional variance of the risk  $X$ , i.e.,

$$TV_q(X) = Var[X | X > x_q] \quad (4.16)$$

Consider an  $n$  random variables  $X_1, X_2, \dots, X_n$ , where each random variable  $X_i$  represents a risk associated with  $i$ -th business line of an insurance company or a loss from the  $i$ -th asset in a portfolio of investment for an individual or an enterprise. The aggregate risk or loss is defined by the sum

$$S = \sum_{i=1}^n X_i.$$

Furman and Landsman (2006) [25] defined the Tail Covariance Premium (TCovP) as

$$TCovP_q(X_i | S) = TVaR_q(X_i | S) + aTCov_q(X_i | S), \quad (4.17)$$

where  $a$  is some non-negative constant,  $i = 1, 2, \dots, n$  and

$$TVaR_q(X_i | S) = E[X_i | S > s_q] \quad (4.18)$$

and Tail Covariance

$$TCov_q(X_i | S) = Cov[X_i, S | S > s_q]. \quad (4.19)$$

In this risk measure the unit of tail covariance is money square. It would be more natural to express it in the unit of money.

Inspired by the allocation from Overbeck (2000) [47],

$$E[X_i] + a \frac{Cov[X_i, S]}{\sigma_S}, \quad (4.20)$$

we define the adjusted risk measure based on the Tail Covariance Premium (TCovP). The adjusted one keeps the money unit.

**Definition 4.2.** *Tail Covariance Premium Adjusted (TCPA) is*

$$TCPA_q(X_i | S) = TVaR_q(X_i | S) + a \frac{TCov_q(X_i | S)}{\sqrt{TV_q(S)}}, \quad (4.21)$$

where

$$TV_q(S) = Var[S | S > s_q]. \quad (4.22)$$

Furman and Landsman (2006) [25] have proved that

$$\sum_{i=1}^n TVaR_q(X_i | S) = TVaR_q(S), \quad (4.23)$$

and

$$\sum_{i=1}^n TCov_q(X_i | S) = TV_q(S). \quad (4.24)$$

So it is straightforward to obtain that

$$\sum_{i=1}^n TCPA_q(X_i | S) = TVaR_q(S) + a \sqrt{TV_q(S)} = TSDP_q(S). \quad (4.25)$$



The sum is exactly the Tail Standard Deviation Premium (TSDP) defined by Furman and Landsman (2006) [25].

Denote the capital allocation proportion to the business line  $i$  by  $k_i$ , then  $k_i$

$$k_i = \frac{TCPA_q(X_i | S)}{TVP_q(S)} = \frac{TVaR_q(X_i | S) + a \frac{TCov_q(X_i | S)}{\sqrt{TV_q(S)}}}{TVaR_q(S) + a\sqrt{TV_q(S)}}. \quad (4.26)$$

The allocation principle is additivity.

The advantage of this method is considering both expectation and standard deviation on tails. The total sum is a money unit.

The article, Wang (2012) [70] considers two business lines with the exponential loss distributions linked by a Farlie-Gumbel- Morgenstern (FGM) copula, modelling the dependence between them. As allocation principle we use the Tail Covariance Premium Adjusted (TCPA) and obtain expressions for the allocation to the two business lines.

### 4.3.1 Calculation of the capital allocation

In this subsection, the capital allocation based the Tail Covariance Premium Adjusted (TCPA), see the equation (4.26), will be calculated. Bargès et al. (2009) [5] have calculated  $TVaR_q(X_i | S)$  and  $TVaR_q(S)$  and the results are presented in equation (4.27) and (4.28). The proofs can be found from section 3.1 in Bargès et al. (2009) [5]. After that, we will calculate the other parts of TCPA, such as  $TCov_q(X_i | S)$  and  $TV_q(S)$ .

Let  $X_1$  and  $X_2$  be two exponentially distributed random variables with parameters  $\lambda_1$  and  $\lambda_2$ , respectively, and the dependence is defined by the FGM copula with parameter  $\theta$ . Then the  $TVaR$  of the aggregate risk  $S = X_1 + X_2$  at level  $q$ ,  $q \in (0, 1)$ , is

$$TVaR_q(S) = \frac{(1+\theta)\zeta(s_q;\lambda_1;\lambda_2) - \theta\zeta(s_q;2\lambda_1;\lambda_2) - \theta\zeta(s_q;\lambda_1;2\lambda_2) + \theta\zeta(s_q;2\lambda_1;2\lambda_2)}{1-q}, \quad (4.27)$$

where  $\zeta(x; \gamma_1; \gamma_2) = \frac{\gamma_2}{\gamma_2 - \gamma_1} e^{-\gamma_1 x} (x + \frac{1}{\gamma_1}) + \frac{\gamma_1}{\gamma_1 - \gamma_2} e^{-\gamma_2 x} (x + \frac{1}{\gamma_2})$ .

Then the  $TVaR$ -based contribution of risk  $i$ ,  $i = 1, 2$ , to the aggregate risk  $S = X_1 + X_2$

at level  $q$ ,  $q \in (0, 1)$ , is

$$TVaR_q(X_i|S) = \frac{(1+\theta)\xi(s_q; \lambda_i; \lambda_j) - \theta\xi(s_q; 2\lambda_i; \lambda_j) - \theta\xi(s_q; \lambda_i; 2\lambda_j) + \theta\xi(s_q; 2\lambda_i; 2\lambda_j)}{1-q}, \quad (4.28)$$

where  $\xi(x; \gamma_i; \gamma_j) = \frac{\gamma_j}{\gamma_j - \gamma_i} e^{-\gamma_i x} \left(x + \frac{1}{\gamma_i}\right) - \frac{\gamma_j e^{-\gamma_i x} - \gamma_i e^{-\gamma_j x}}{(\gamma_j - \gamma_i)^2}$  and  $i \neq j$ .

The Tail covariance  $TCov_q(X_i | S)$  is given in the following proposition.

**Proposition 4.2.** *Let  $X_1$  and  $X_2$  be two exponentially distributed random variables with dependence defined by the FGM copula. Then the  $TCov_q(X_i | S)$  of risk  $i$ ,  $i = 1, 2$  is*

$$\begin{aligned} & TCov_q(X_i|S) \\ &= \frac{1}{1-q} \left( (1+\theta)\varsigma(s_q; \lambda_i; \lambda_j) - \theta(\varsigma(s_q; 2\lambda_i; \lambda_j) + \varsigma(s_q; \lambda_i; 2\lambda_j) - \varsigma(s_q; 2\lambda_i; 2\lambda_j)) \right) \\ & \quad - TVaR_q(S)TVaR_q(X_1|S), \end{aligned} \quad (4.29)$$

where

$$\varsigma(x; \gamma_i; \gamma_j) = \frac{\gamma_j \left( x^2 + \frac{2x}{\gamma_i} + \frac{2}{\gamma_i^2} \right) e^{-\gamma_i x}}{\gamma_j - \gamma_i} - \frac{\gamma_j \left( x + \frac{1}{\gamma_i} \right) e^{-\gamma_i x} - \gamma_i \left( x + \frac{1}{\gamma_j} \right) e^{-\gamma_j x}}{(\gamma_j - \gamma_i)^2}, \quad (4.30)$$

$i \neq j$ ,  $TVaR_q(S)$  and  $TVaR_q(X_i|S)$  from (4.27) and (4.28).

**Proposition 4.3.** *Let  $X_1$  and  $X_2$  be two exponentials with dependence defined by the FGM copula. Then the  $TV_q(S)$  is*

$$TV_q(S) = \sum_{i=1}^2 TCov_q(X_i | S), \quad (4.31)$$

where  $TVaR_q(X_i | S)$  are obtained from (4.28).

**Proposition 4.4.** *Let  $X_1$  and  $X_2$  be two exponentials with dependence defined by the FGM copula. Then the  $TCPA_q(X_i | S)$  of risk  $i$ ,  $i = 1, 2$  is*

$$TCPA_q(X_i | S) = TVaR_q(X_i | S) + a \frac{TCov_q(X_i | S)}{\sqrt{TV_q(S)}}, \quad (4.32)$$

where  $a$  is some non-negative constant, and  $TVaR_q(X_i | S)$ ,  $TCov_q(X_i | S)$  and  $TV_q(S)$  are obtained from (4.28), (4.29) and (4.31), respectively.

*Remark 4.1.* Like (4.25), the Tail Variance Premium  $TVP$  of  $S$  is the sum of the  $TCPA$  of  $X_i$ ,  $i = 1, 2$

$$\sum_{i=1}^2 TCPA_q(X_i | S) = TVaR_q(S) + a\sqrt{TV_q(S)} = TVP_q(S). \quad (4.33)$$

# Chapter 5

## Illustrative case study

This chapter will present a case from Valdez (2012) [65].

For purpose of showing illustrations, we will consider an insurance company with five lines of business:

- auto insurance - property damage
- auto insurance - liability
- household or home-owners' insurance
- professional liability
- other lines of business

We will measure loss on a per premium basis, i.e. loss ratio and denote the random variable by  $S$  for the entire company and  $X_i$  for the  $i$ -th line of business,  $i = 1, 2, 3, 4, 5$ .

We assume that the loss ratio distributions of different lines of business follow in the Table 5.1 and the dependence between lines of business in the correlation matrix.

TABLE 5.1: The distributions of lines of business.

Line of business	Loss ratio distribution	Premium share $w_i$	Parameters	Mean	Variance
Auto(PD)	Gamma	30%	$\alpha = 360, \beta = 600$	0.60	0.001
Auto(liab)	Lognormal	20%	$\mu = -0.362, \sigma = 0.101$	0.70	0.005
Household	Gamma	15%	$\alpha = 56.25, \beta = 75$	0.75	0.01
Prof liab	Pareto	15%	$a = 6.92, b = 4.74$	0.8	0.9
Other	Lognormal	20%	$\mu = -0.784, \sigma = 0.427$	0.5	0.05

$$Cor = \begin{matrix} & auto(PD) & Auto(liab) & Household & Profliab & Other \\ \begin{matrix} Auto(PD) \\ Auto(liab) \\ Household \\ Profliab \\ Other \end{matrix} & \left( \begin{matrix} 1 & & & & \\ 0.4 & 1 & & & \\ 0.1 & 0.1 & 1 & & \\ 0.2 & 0.5 & 0.1 & 1 & \\ 0.05 & 0.2 & 0.1 & 0.4 & 1 \end{matrix} \right) & \end{matrix} \quad (5.1)$$

The density functions of these distributions are in the table 5.2.

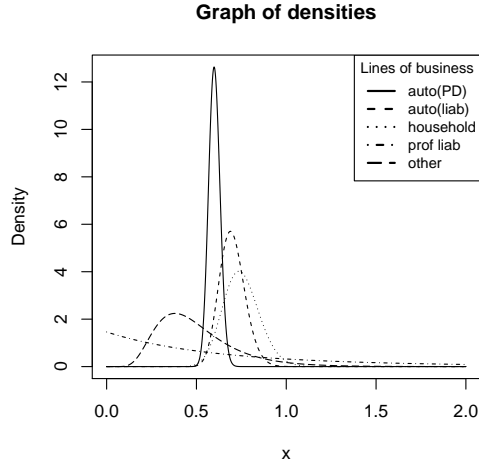
TABLE 5.2: The pdf, quantile and CTE for distributions of lines of business.

Distribution	density $f_X(x)$	Quantile $Q_p(X)$	$CTE_p(X)$
Gamma	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	no explicit form	$\frac{F_X(x_p; \alpha+1, \beta)}{F_X(x_p; \alpha, \beta)} \frac{\alpha}{\beta}$
Lognormal	$\frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}}$	$e^{\mu+\phi^{-1}(p)\sigma}$	$\frac{\phi(\sigma-\phi^{-1}(p))}{1-p} e^{\mu+\frac{\sigma^2}{2}}$
Pareto	$\frac{ab^a}{(x+b)^{a+1}}$	$b[(1-p)^{-\frac{1}{a}} - 1]$	$\frac{a}{a-1} Q_p[x] + \frac{b}{a-1}$

We can see the pdf of lines of business in the figure 5.1.

We use the same method to simulate as one in the appendix from Tang and Valdez (2009) [53].

FIGURE 5.1: The pdf of lines of business.



## 5.1 Simulation from Normal copula

The following algorithm simulate  $n$  observations of the loss ratios for each business lines from normal copula.

1. Generate a data set with 5 columns of standard normal random variables  $Y = (Y_1, Y_2, Y_3, Y_4, Y_5)$  with correlation matrix  $Cor$ ;
2. Set  $U_i = \Phi(Y_i)$ ;
3. Set  $X_i = F_{X_i}^{-1}(U_i)$ ;

For the step 1, it is not difficult to find some functions to simulate in some languages, for example, R. We can obtain the random variables  $Y$  as following:

- Construct the lower triangular matrix  $L$  so that the covariance matrix  $Cor = LL^T$  using Choleski's decomposition;

- Generate a column vector of independent standard normal random variables  $Z = (Z_1, Z_2, \dots, Z_n)^T$ ;
- Take the matrix product of  $L$  and  $Z$ , i.e.  $Y = LZ$ .

Using the procedure as outlined above, we generate 10000 observations of the loss ratios for each business line.

Table 5.3 provides stand-alone capitals for each business line. Only the line of business, Professional liability has quite different numbers for different allocation rules. The main reason is that the loss of this line of business has heavier tail comparing other lines of business. The last line is the weighted sum by premium share and the weight  $W = (30\%, 20\%, 15\%, 15\%, 20\%)$  are from Table 5.1. The weighted sum of  $TVaR_q(X_i|S)$  and  $TCPA_q(X_i|S)$  are  $TVaR_q(S)$  and  $TVP_q(S)$  respectively.

TABLE 5.3: Some risk measure of loss distributions,  $q = 0.995$

Line of business	$VaR_{0.995}(X_i)$	$TVaR_{0.995}(X_i)$	$TVaR_{0.995}(X_i S)$	$TCPA_{0.995}(X_i S)$
Auto (PD)	0.6853	0.6973	0.6182	0.6187
Auto(liab)	0.8995	0.9282	0.8078	0.8110
Household	1.0297	1.0710	0.7798	0.7802
Prof liab	5.6950	7.3293	7.2563	7.6997
Other	1.3235	1.4917	0.9321	0.9347
Weighted sum by premium share	1.6589	1.9532	1.7389	1.8067
			$TVaR_q(S)$	$TVP_q(S)$

Note: we can see in the table,  $TVaR_q(X_i|S)$  is smaller than the corresponding  $TVaR_q(X_i)$ . The reason is that  $TVaR_q(X_i)$  is the mean of the top  $(1 - q)$  of  $X_i$ , and  $TVaR_q(X_i|S)$  is the mean of  $X_i$  when the sum  $S$  is on the top  $(1 - q)$ . So the  $X_i$  from  $TVaR_q(X_i|S)$  can not be greater than the  $X_i$  from  $TVaR_q(X_i)$ , at least the same.

Table 5.4 provides some important summary statistics of the aggregate loss distribution. We can see the  $VaR_{0.995}(S)$ , 1.4795 is smaller than the weighted sum  $\sum w_i VaR_{0.995}(X_i)$ , 1.6589. In this case, the VaR is subadditive.

TABLE 5.4: Summary statistics of the aggregate loss distribution

Statistic	
Mean	0.6512
SD	0.1736
Mode	0.5704
Skewness	2.7361
Kurtosis	20.2716
Minimum	0.3967
1st quantile	0.5382
Median	0.6077
3rd quantile	0.7148
Maxim	3.5414
$VaR_{0.995}(S)$	1.4795
$TVaR_{0.995}(S)$	1.7389
$TVP_{0.995}(S)$	1.8067

### 5.1.1 Proportional capital allocation

Table 5.5 is the comparing result of proportional allocations when  $\rho(X_i, S)$  takes different risk measures,  $VaR_q(X_i)$  and  $TCPA_q(X_i|S)$  respectively. We calculated the allocation percentages by  $\rho(X_i, S) / \sum_{i=1}^n \rho(X_i, S)$ , then times the capital  $K = VaR_q(S)$  to generate the allocation amounts to different business lines.

TABLE 5.5: Proportional allocation when the risk measure  $\rho(X_i, S)$  is  $VaR_q(X_i)$  and  $TCPA_q(X_i|S)$  respectively,  $q = 0.995$ .

$\rho(X_i, S) =$	$VaR_q(X_i)$		$TCPA_q(X_i S)$	
	Percentage %	allocation	Percentage %	allocation
Auto (PD)	12.4	0.1833	10.3	0.1520
Auto(liab)	10.9	0.1605	9.0	0.1328
Household	9.3	0.1378	6.5	0.0958
Prof liab	51.5	0.7619	63.9	0.9458
Other	16.0	0.2361	10.3	0.1531
Total $VaR_q[S]$		1.4795		1.4795



### 5.1.2 Covariance capital allocation

For the covariance capital allocation, the covariance matrix is

$$\begin{array}{l}
 \begin{array}{c}
 \textit{auto}(PD) \\
 \textit{Auto}(liab) \\
 \textit{Household} \\
 \textit{Profliab} \\
 \textit{Other}
 \end{array}
 \begin{pmatrix}
 \textit{auto}(PD) & \textit{Auto}(liab) & \textit{Household} & \textit{Profliab} & \textit{Other} \\
 \left( \begin{array}{ccccc}
 9 & 5 & 1 & 27 & 2 \\
 5 & 20 & 2 & 101 & 13 \\
 1 & 2 & 23 & 21 & 7 \\
 27 & 101 & 21 & 2025 & 255 \\
 2 & 13 & 7 & 255 & 200
 \end{array} \right)
 \end{pmatrix}
 10^{-5},
 \end{array}$$

where the  $(i, j)$ -th element is the covariance  $Cov[w_i X_i, w_j X_j]$ . The allocation percentage  $Cov[w_i X_i, S]/Var[S]$  is the sum of the appropriate row of the matrix divided by the sum of all elements. The allocation percentages for the business lines respectively are (1.4%, 4.5%, 1.7%, 77.2%, 15.1%). The corresponding allocation amounts are (0.0211, 0.0662, 0.0255, 1.1427, 0.2240).

## 5.2 Simulation from Cauchy copula

We generate 10,000 observations of the loss distributions for each business line from Cauchy copula. The method of generating can be found in the appendix in Tang and Valdez (2009) [53], the last part, Cauchy copula.

1. Generate a data set with 5 columns of standard normal random variables  $Y = (Y_1, Y_2, Y_3, Y_4, Y_5)$  with correlation matrix  $Cor$ ;
2. Generate a chi-squared random variable  $S \sim \chi^2(1)$  with 1 degree of freedom;
3. Set  $T_i = \sqrt{1/S} Y_i$
4. Set  $U_i = t_1(T_i)$ ;
5. Set  $X_i = F_{X_i}^{-1}(U_i)$ ;

Table 5.6 firstly provides some risk measures of loss distributions without considering premium share. From the table, we can see that only the line of business, Prof liab has quite different numbers for different allocation rules. The main reason is that the loss of this line of business has heavier tail comparing other lines of business. For the business line, Household, the difference of  $TVaR[X_i|S]$  and  $TVaR[X_i]$  is comparing greater. The reason may be that the correlations between Household and other business lines are small. Then the last line is the weighted sum by premium share, see the weight  $W=(30\%, 20\%, 15\%, 15\%, 20\%)$  from Table 5.1. The weighted sum of  $TVaR_q[X_i|S]$  and  $TCPA_q[X_i|S]$  are  $TVaR_q[S]$  and  $TVP_q[S]$  respectively.  $TVaR_q[S]$  is 1.7290 and  $TVP_q[S]$  is 1.7701.

TABLE 5.6: Some risk measures of loss distributions,  $q = 0.995$

Line of business	$VaR_q[X_i]$	$TVaR_q[X_i]$	$TVaR_q[X_i S]$	$TCPA_q[X_i S]$
Auto (PD)	0.6851	0.6933	0.6091	0.6089
Auto(liab)	0.8987	0.9251	0.8361	0.8397
Household	1.0375	1.0740	0.8641	0.8748
Prof liab	5.2863	6.8671	6.7211	6.9396
Other	1.3407	1.4837	1.2061	1.2364
Weighted sum by premium share	1.6020	1.8809	1.7290 $TVaR_q[S]$	1.7701 $TVP_q[S]$

The  $VaR_{0.995}[S]$  from the simulation data is 1.4899. We can see that it is smaller than the weighted sum  $\sum w_i VaR_{0.995}[X_i]$ , 1.6020. In this case, the VaR is subadditive,  $VaR_{0.995}[\sum w_i X_i] < \sum w_i VaR_{0.995}[X_i]$ .

### 5.2.1 Proportional capital allocation

Table 5.7 is the comparing result of proportional allocations when  $\rho[X_i, S]$  takes different risk measures,  $VaR_q[X_i]$  and  $TCPA_q[X_i|S]$  respectively. We calculated the allocation percentages by  $\rho[X_i, S] / \sum_{i=1}^n \rho[X_i, S]$ , then times the capital  $K = VaR_q[S]$  to generate the allocation amounts to different business lines.

TABLE 5.7: Proportional allocation when the risk measure  $\rho[X_i, S]$  is  $VaR_q[X_i]$  and  $TCPA_q[X_i|S]$  respectively,  $q = 0.995$ .

$\rho[X_i, S] =$	$VaR_q[X_i]$		$TCPA_q[X_i S]$	
	Percentage %	allocation	Percentage %	allocation
Auto (PD)	12.8	0.1912	10.3	0.1538
Auto(liab)	11.2	0.1672	9.5	0.1414
Household	9.7	0.1447	7.4	0.1105
Prof liab	49.5	0.7375	58.8	0.8762
Other	16.7	0.2494	14.0	0.2081
Total $VaR_q[S]$		1.4899		1.4899

## 5.2.2 Covariance capital allocation

For the covariance capital allocation, the allocation percentages  $Cov[w_i X_i, S]/Var[S]$  keep the same (1.4%, 4.5%, 1.7%, 77.2%, 15.1%), see the normal copula case. The correspond allocation amounts are (0.0213, 0.0667, 0.0256, 1.1507, 0.2256).

# Chapter 6

## Conclusion

The purpose of this thesis is to give a broad self-contained overview of risk measures and capital allocation, in particular, allocation based on heavy-tailed risks.

In chapter 1, we discuss the concept of economic capital, the need for capital allocation, different viewpoints and give an overview of the literature.

In chapter 2, basic concepts are presented. In the part on risk measures, we define coherent risk measures and review a number of important risk measures used.

In chapter 3, we go through different principles of capital allocation and we choose to use the proportional capital allocation.

In chapter 4, since our focus is on insurance, especially insurance products with heavy-tailed distributions, we emphasize capital allocation based on tails. We define the concepts of Conditional Expectation, Upper Tail Covariance and Tail Covariance Premium Adjusted. Finally, in chapter 5, a simulation study is made for a portfolio with different insurance products.

In this thesis, we want to state that one single allocation principle is not perfect for all purposes. We adopt the point of view of insurance policyholders and regulators. The risk capital should be sufficient to compensate clients with different kinds of insurance policies in difficult, even catastrophic situations. This approach will rule out allocation only based on covariance, since covariance allocation allocates nothing at all to no-risk business lines. The risk measure Tail Covariance Premium Adjusted (TCPA) includes TVaR and Tail

covariance. It is suited for heavy-tailed distributions. Thus the allocation principle based on TCPA turns out to be well suited.

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